On statistical problems related to navigation with LEO satellites

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Workshop in Honor of Yury Kutoyants 80th Birthday
Le Mans, 17–18 September, 2026

The emergence of Low Earth Orbit (LEO) satellite constellations designed to provide internet access naturally leads to using these satellites for navigation.
The idea of determining the receiver location on the Earth surface is based on computing Doppler's shift profiles for several satellites.


Psiaki M. (2021) *Navigation using carrier Doppler shift from a LEO constellation: TRANSIT on steroids.* Navigation - Journal of The Institute of Navigation. **68**, 3, 621–641.

This talk considers some simplified statistical problems related to navigation using the Starlink satellites.


Humphreys T. E., Iannucci P. A., Komodromos Z. M., Graff A. M. *Signal Structure of the Starlink Ku-Band Downlink.* IEEE Transactions on Aerospace and Electronic Systems. 2023, V. 59, No.5, P. 6016 – 6030.

I would like to show how these problems are related to classical approaches in


Ibragimov I.A., Khasminskii R.Z. (1981) *Statistical Estimation. Asymptotic Theory.* Springer New York.


Kutoyants Yu. (2013) *An introduction to the statistics of Poisson processes and applications.* Springer International Publishing AG.

From a mathematical viewpoint the problem is to recover an unknown smooth function

$$\theta'(t) = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t},\ t \in [-T/2, T/2],$$

from the noisy observations

$$\mathrm{d}Y(t) = X[t - \theta(t)]\mathrm{d}t + \sigma \mathrm{d}\widetilde{W}(t),\ \ t \in [-T/2, T/2],$$

where

- $\widetilde{W}(\cdot) \in \mathbb{C}$ is two-sided complex valued Wiener's process

$$\mathbf{E}\{\widetilde{W}(t)\widetilde{W}^*(s)\} = \frac{|t| + |s| - |t - s|}{2}.$$

- $X(\cdot) \in \mathbb{C}$ is an OFDM random process.

In order to simplify numerous technical details it is assumed that

$$\theta(t) = \theta + \theta' t,$$

where

- the offset time $\theta \in \mathbb{R}$,
- the Doppler's shift

$$\theta' = \frac{\text{satellite's radial velocity}}{\text{speed of light}}, \quad |\theta'| \ll 1.$$

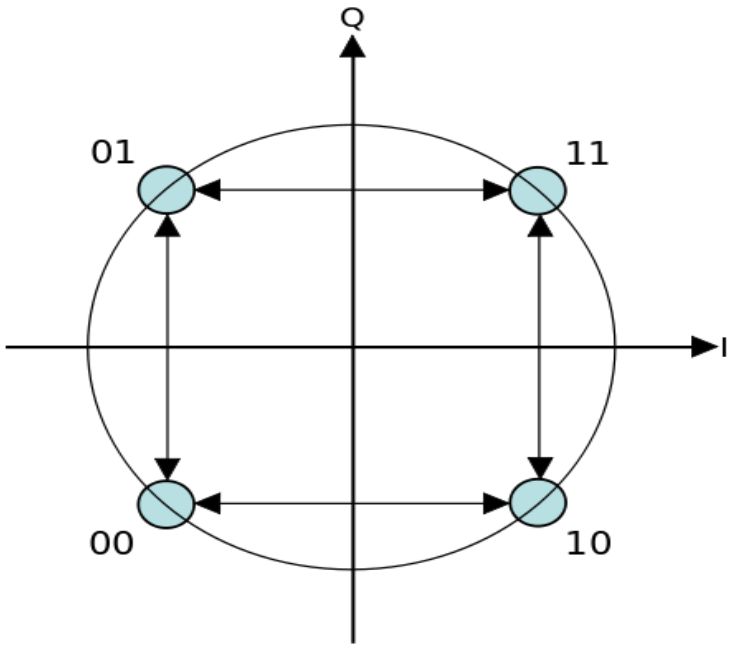
Orthogonal frequency-division multiplexing (OFDM)

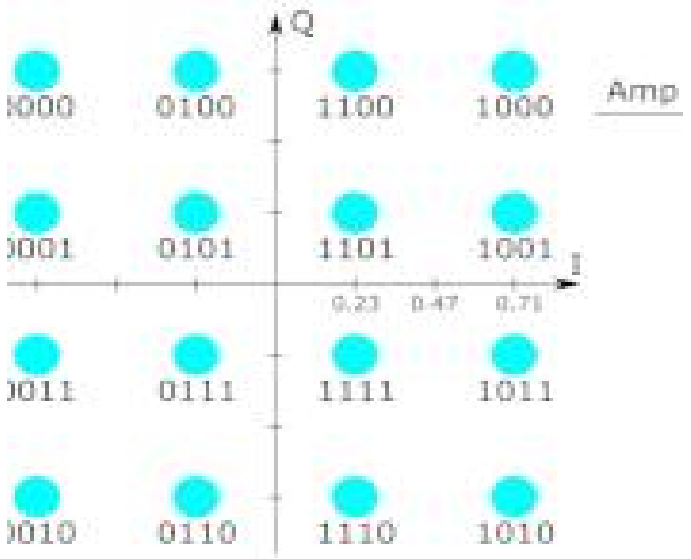
OFDM is used for encoding digital (binary) data on multiple carrier frequencies.
OFDM signal with N sub-carriers is defined by

$$X(t) = \frac{1}{\sqrt{N}} \sum_{k=-N/2}^{N/2} x_{k,s} \exp(2\pi \mathrm{i} k t), t \in [s - 0.5, s + 0.5],$$

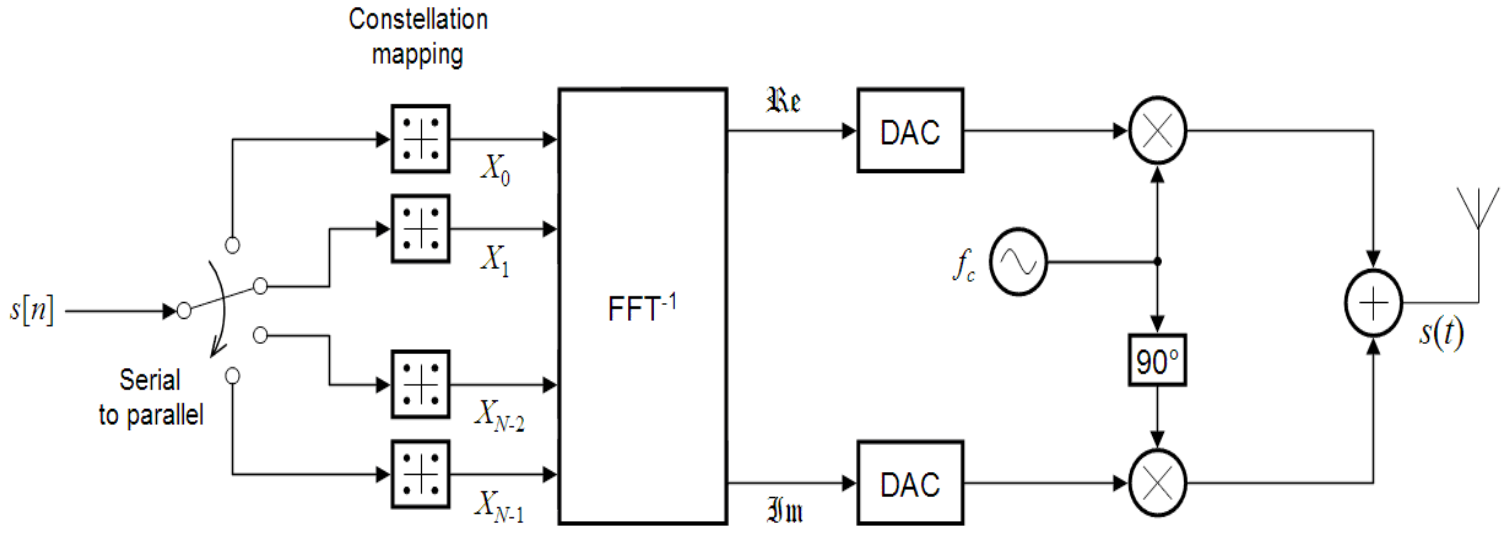
where $x_{k,s} \in \mathbb{C}$ encodes bits of information depending on the modulation scheme.

Starlink uses the OFDM with $N = 1024$ or $N = 2048$ sub-carriers and 4QAM (Quadrature Amplitude Modulation) or 16QAM modulation schemes:



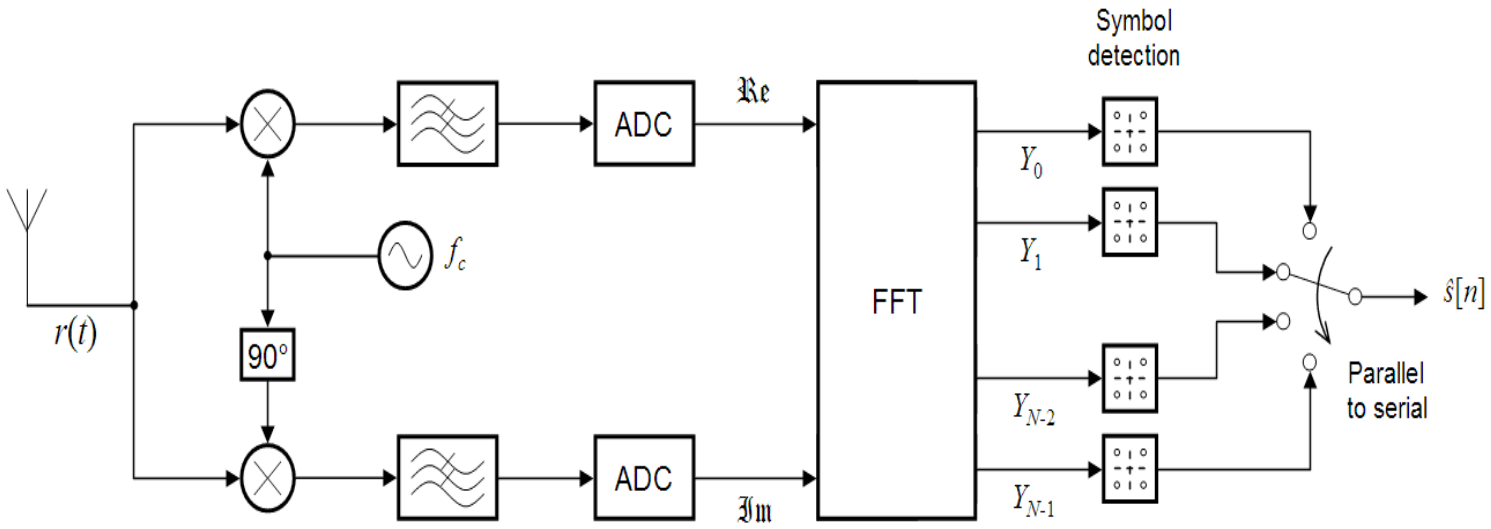


Ideal OFDM transmitter



https://en.wikipedia.org/wiki/Orthogonal_frequency-division_multiplexing

Ideal OFDM receiver



https://en.wikipedia.org/wiki/Orthogonal_frequency-division_multiplexing

Without loss of generality we can assume that for given s symbols $x_{k,s}$ are independent random variables with

$$\mathbf{E}\{x_{k,s}\} = 0, \ \ k = -N/2, \ldots, N/2.$$

The distance between the receiver and the LEO satellite changes significantly. Therefore in order to provide reliable demodulation, periodic insertions are embedded is the transmitted signal , i.e.,

$$x_{k,m\tau} = x_k^{\circ}, \ \ m = 0, \pm 1, \ldots.$$

As to

$$x_{k,s}, s \neq 0, \pm\tau, \pm 2\tau, \ldots$$

they are independent.

Demodulation of s-th ODFDM symbol is based on ADC outputs

$$\bar{Y}_{sN+k} \approx N \int_{s+k/N-1/(2N)}^{s+k/N+1/(2N)} \mathrm{d}Y(t), \;\; k \in [-N/2, N/2].$$

Since N is large, in view of the central limit theorem we have the following simple probabilistic model

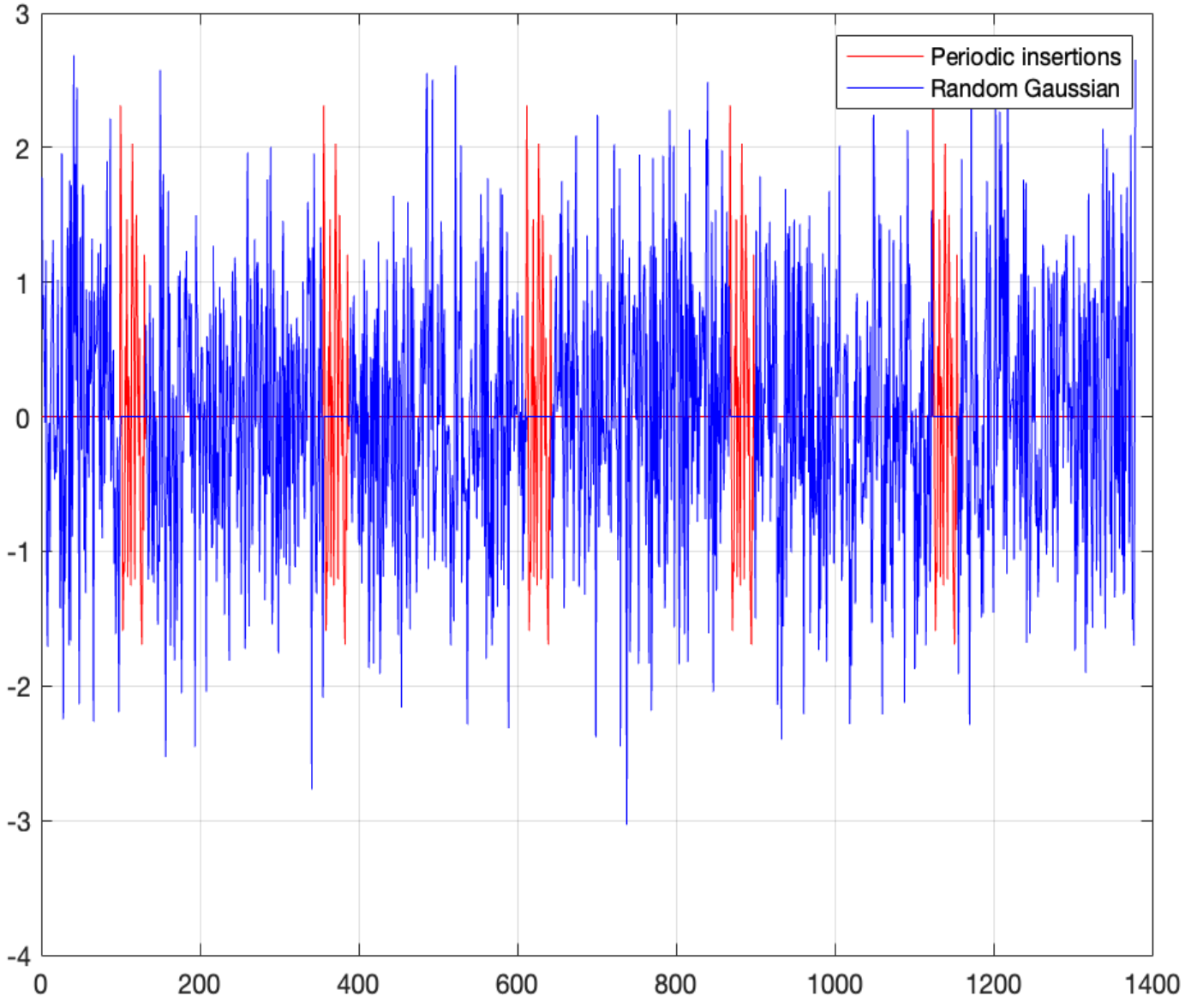
$$\bar{Y}_{sN+k} \approx \bar{X}_{sN+k-(\theta' s+\theta)N} + \sigma\sqrt{N}\,\xi_{sN+k}, \;\; k \in [-N/2, N/2],$$

where $\xi_p, \;\; p \in \mathbb{Z}$, are independent complex-valued $\mathcal{N}(0,1)$ random variables.

- $$\bar{X}_{sN+k} = \mu_k, \;\; s = 0, \pm\tau, \pm 2\tau, \ldots, \;\; k \in [-N/2, N/2],$$

  where μ_m are i.i.d. complex-valued $\mathcal{N}(0,1)$,
- $$\bar{X}_{sN+k}, \;\; s \neq 0, \pm\tau, \pm 2\tau, \ldots, \;\; k \in [-N/2, N/2],$$

  are i.i.d. complex-valued $\mathcal{N}(0,1)$.
- $\mu_k \sim \mathcal{N}(0,1)$ are i.i.d. unknown nuisance parameters.


Periodic insertions
Random Gaussian
3
2
1
0
-1
-2
-3
-4
0
200
400
600
800
1000
1200
1400

Offset time estimation

Assume that $\theta' = 0$. Our the goal is to estimate the off-set time θ based on

$$\bar{Y}_{sN+k} = \bar{X}_{sN+k-\theta N} + \sigma\sqrt{N}\xi_{sN+k},$$

where $k \in [-N/2, N/2]$, $s = [-T/2, T/2]$.
With the simple identity

$$\max_{\mu}\left\{-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left|Y_i - \mu\right|^2 - |\mu|^2\right\}$$
$$= -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left|Y_i\right|^2 + \frac{1}{\sigma^2/n(1+\sigma^2/n)}\left|\frac{1}{n}\sum_{i=1}^{n}Y_i\right|^2.$$

we arrive at the following formula for the log of posterior probability

$$L\left(\theta; \bar{\mathbf{Y}}^T\right) = \frac{1}{\sigma_{NT}^2(1+\sigma_{NT}^2)} \sum_{|k|\leq N/2} \left| \frac{\tau}{T} \sum_{|p|\leq T/(2\tau)} \bar{Y}_{\theta N - k + p\tau} \right|^2 + K,$$

where K is a constant that doesn't depend on θ and

$$\sigma_{NT}^2 \stackrel{\text{def}}{=} \frac{\sigma^2 N\tau}{T}.$$

Let $\theta_\circ$ be the true value of θ and $\mathbf{P}_{\theta_\circ}$ is the corresponding distribution of $\bar{\mathbf{Y}}^T$.

Lemma

Uniformly in $\theta \in (\theta_\circ, \theta_\circ + 0.25)$

$$L\left(\theta; \bar{\mathbf{Y}}^T\right) - L\left(\theta_\circ; \bar{\mathbf{Y}}^T\right)$$

$$\overset{\mathbf{P}_{\theta_\circ}}{=} \frac{\sigma_{NT}^2 + \tau/T}{\sigma_{NT}^2(1+\sigma_{NT}^2)} \sum_{i=1}^{(\theta-\theta_\circ)N} \chi_i - \frac{1}{\sigma_{NT}^2} \sum_{i=1}^{(\theta-\theta_\circ)N} \chi_i',$$

where χ_i, χ_i' *are i.i.d. standard exponentially distributed r. v.*

Main conditions

In the following it is assumed that N is a large parameter i.e. $N \to \infty$.

It is assumed also that σ, τ, T depend on N and

- $$\lim_{N\to\infty} \frac{T_N}{\tau_N} = \infty; \tag{1}$$
- small sample size condition

$$\lim_{N\to\infty} \sigma_{NT}^2 = \lim_{N\to\infty} \frac{\sigma_N^2 N}{T_N/\tau_N} = \infty; \tag{2}$$

- consistency condition

$$\lim_{N\to\infty} \frac{\sigma_{NT}^2}{\sqrt{N}} = \lim_{N\to\infty} \frac{\sigma_N^2 \sqrt{N}}{T_N/\tau_N} = 0. \tag{3}$$

Roughly speaking, we assume

$$\sigma_N^2 \sqrt{N} \ll \frac{T_N}{\tau_N} \ll \sigma_N^2 N.$$

Theorem

Under conditions 1-3, as $N \to \infty$

$$L\left(\theta + r_{NT} t; \bar{\mathbf{Y}}^T\right) - L\left(\theta; \bar{\mathbf{Y}}^T\right) \xrightarrow{\mathbf{P}_\theta} -\frac{|t|}{2} + \widetilde{W}(t), \ |t| \leq K,$$

where

$$r_{NT} \stackrel{\text{def}}{=} \frac{\sigma_{NT}^4}{2N} = \frac{\sigma_N^4 N \tau_N^2}{2 T_N^2}.$$

Let $\widehat{\theta}^T$ be the maximum posterior probability estimate of θ

$$\widehat{\theta}^T = \arg\max_{\theta} L\big(\theta; \bar{\mathbf{Y}}^T\big).$$

Theorem

If (1)-(3) *hold true and for some* $\epsilon > 0$

$$\lim_{N\to\infty}\left[\frac{1}{8r_{NT}} - \big(1+\epsilon\big)\log(\tau_N)\right] = \infty,$$

then for all $|\theta| \leq \tau_N/2 - \epsilon$

$$\lim_{N\to\infty} \mathbf{P}_\theta\Big\{r_{NT}^{-1}|\widehat{\theta}^T - \theta| > x\Big\} = \mathbf{P}\big\{|\eta| > x\big\},$$

where

$$\eta = \arg\max_{t\in\mathbb{R}}\left[-\frac{|t|}{2} + \widetilde{W}(t)\right].$$

This limit theorem looks similar to the results that can be found in

- Ibragimov I.A., Khasminskii R.Z. (1981) *Statistical Estimation: Asymptotic Theory.* Springer-Verlag, New York.
- Kutoyants Yu. (2013) *An introduction to the statistics of Poisson processes and applications.* Springer International Publishing AG.

The Laplace transform of the distribution $|\eta|$ is well known

$$\Psi(z) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{-zx}\,\mathrm{d}\mathbf{P}\{8|\eta| \le x\}$$
$$= \frac{(5+\sqrt{1+z})}{(1+\sqrt{1+z})^2(3+\sqrt{1+z})}.$$

With this formula we get

1. $$\mathbf{E}\eta^2 = 26.$$

2. Tchebyshev-Chernoff bound. As $x \to \infty$

$$\log\Big\{\mathbf{P}\big\{|\eta| \geq x\big\}\Big\}$$
$$\leq -\frac{x}{8} - \frac{1}{2}\log\left(\frac{x}{8}\right) + \log\left(\frac{8\sqrt{2\mathrm{e}}}{9}\right) + o(1).$$

Here

$$\log\left(\frac{8\sqrt{2\mathrm{e}}}{9}\right) \approx 0.7288$$

For the fractional derivative of order 3/2 of

$$\Psi(-z) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{zx}\,\mathrm{d}\mathbf{P}\{8|\eta| \le x\} = \frac{(5+\sqrt{1-z})}{(1+\sqrt{1-z})^2(3+\sqrt{1-z})}$$

we have

$$\mathcal{D}^{3/2}\Psi(-z) = \frac{16}{9\sqrt{\pi}(1-z)} + O\left(\frac{1}{\sqrt{1-z}}\right),\ z \to 1.$$

Large deviations of $|\eta|$

With this formula arrive at the following theorem that yields large deviations of $|\eta|$.

Theorem

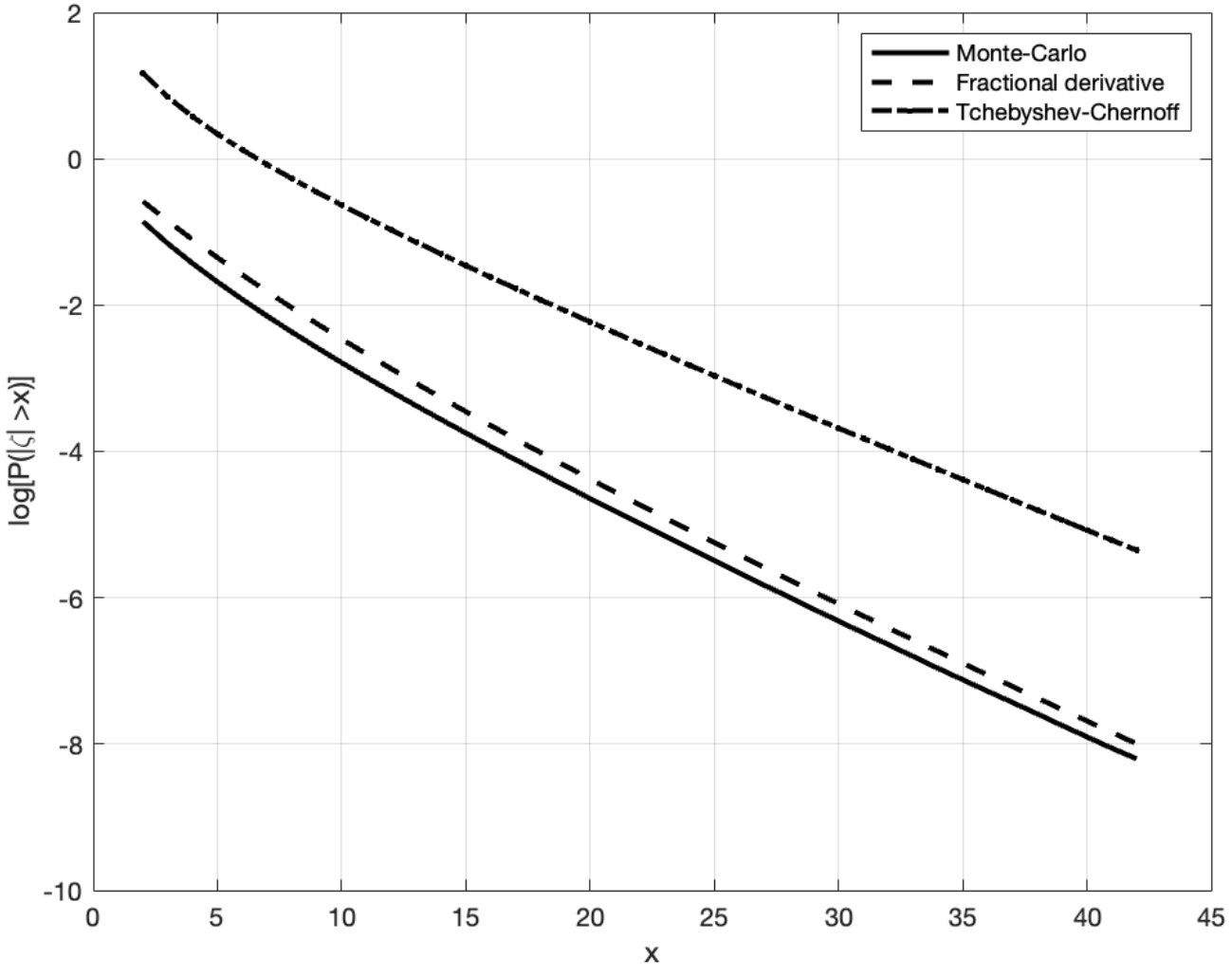
As $x \to \infty$

$$\log\Big\{\mathbf{P}\big\{|\eta| \geq x\big\}\Big\}$$
$$= -\frac{x}{8} - \frac{3}{2}\log\left(\frac{x}{8}+1\right) + \log\left(\frac{16}{9\sqrt{\pi}}\right) + o(1).$$

Notice

$$\log\left(\frac{16}{9\sqrt{\pi}}\right) \approx 0.003.$$

Large deviations of $|\eta|$



Bayes estimates

The MAP estimate $\widehat{\theta}^T$ may be viewed as a particular case $\gamma = \infty$ of

$$\widehat{\theta}_\gamma^T = \int_{-\tau/2}^{\tau/2} \theta \mathrm{e}^{\gamma L(\theta;\bar{\mathbf{Y}}^T)} \mathrm{d}\theta \Big/ \int_{-\tau/2}^{\tau/2} \mathrm{e}^{\gamma L(\theta;\bar{\mathbf{Y}}^T)} \mathrm{d}\theta.$$

It can be shown

$$\lim_{N\to\infty} \mathbf{P}_\theta \Big\{ r_{NT}^{-1} |\widehat{\theta}_\gamma^T - \theta| \geq x \Big\} = \mathbf{P}\Big\{ |\eta_\gamma| > x \Big\},$$

where

$$\eta_\gamma = \frac{\displaystyle\int_{-\infty}^{\infty} t \exp\left\{ \gamma \left[-\frac{|t|}{2} + \widetilde{W}(t) \right] \right\} \mathrm{d}t}{\displaystyle\int_{-\infty}^{\infty} \exp\left\{ \gamma \left[-\frac{|t|}{2} + \widetilde{W}(t) \right] \right\} \mathrm{d}t.}$$

It is known that

1. $$\min_{\gamma>0} \mathbf{E}\left[\eta_\gamma^2\right] = \mathbf{E}\left[\eta_1^2\right].$$

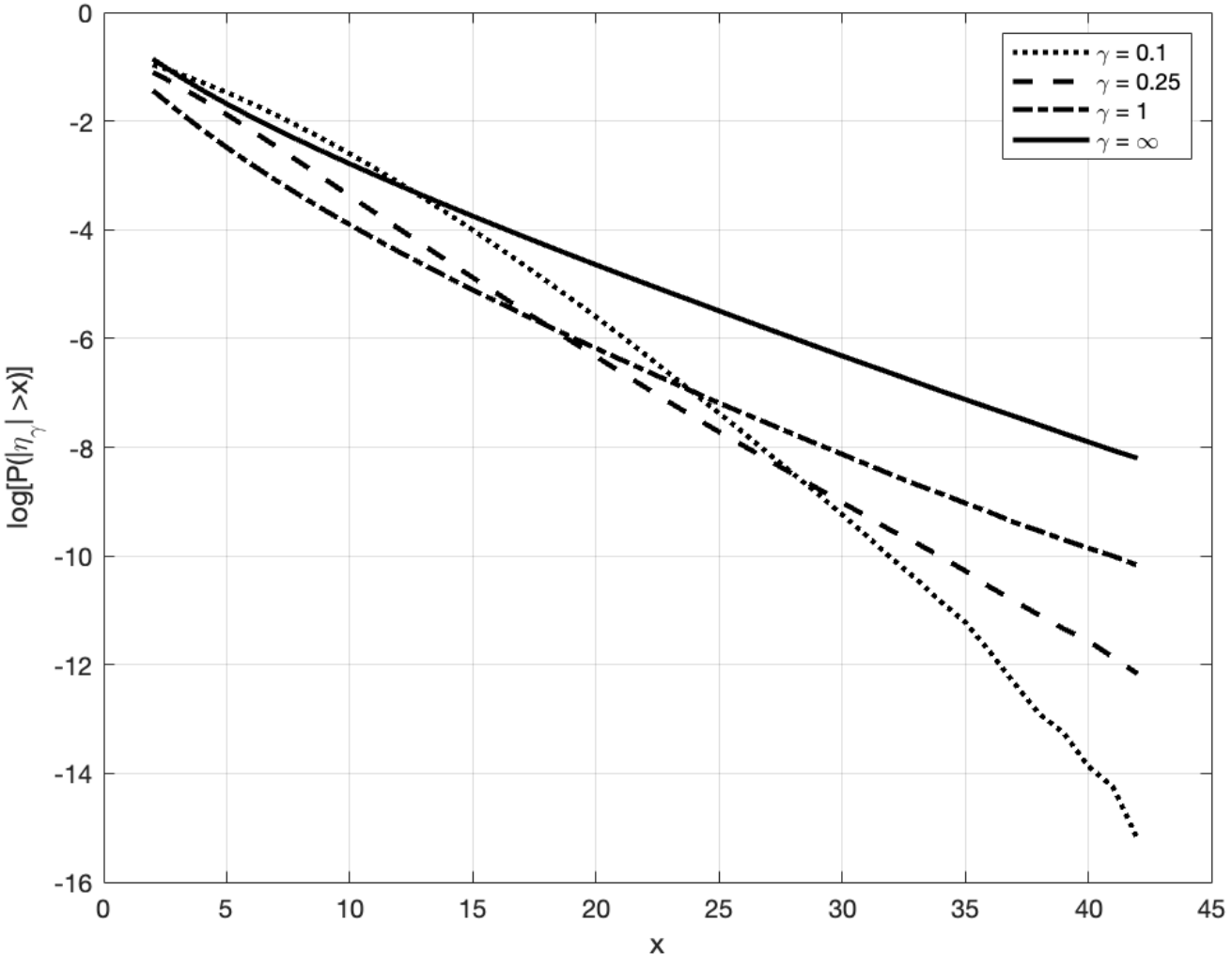
2. $$\mathbf{E}\left[\eta_1^2\right] = 16\zeta(3) \approx 19.2329,$$

   where $\zeta(\cdot)$ is the Riemann zeta function.


Ibragimov I.A., Khasminskii R.Z. (1981) *Statistical Estimation: Asymptotic Theory.* Springer-Verlag, New York.


Rubin H., Song K. *Exact computation of the asymptotic effciency of maximum likelihood estimators of a discontinuous signal in a Gaussian white noise.* Ann. Statist. 1995. V. 23, P. 732–739.

Large deviations of $|\eta_\gamma|$



Equivalent WGN model

We have seen that

$$Z_k = \frac{1}{\sigma_{NT}^2(1+\sigma_{NT}^2)}\left|\frac{\tau}{T}\sum_{|s|\le T/(2\tau)} \bar{Y}_{k+N\tau s}\right|^2,\ k \in [-N\tau/2, N\tau/2].$$

are sufficient statistics for estimating θ.
It easy to check that

$$\begin{aligned} Z_k &\overset{\mathbf{P}_\theta}{=} \chi_k\left\{\frac{\mathbf{1}\{|k-N\theta| \le N/2\}}{\sigma_{NT}^2} + \frac{\mathbf{1}\{|k-N\theta| > N/2\}}{1+\sigma_{NT}^2}\right\} \\ &\overset{\mathbf{P}_\theta}{=} \chi_k\left\{1 + (1+o(1))\frac{\mathbf{1}\{|k-N\theta| \le N/2\}}{\sigma_{NT}^2}\right\},\ N\to\infty. \end{aligned}$$

So for

$$Z'_k = \sigma^2_{NT} \log(Z_k)$$

we get

$$Z'_k \overset{\mathbf{P}_\theta}{=} (1 + o(1))\mathbf{1}\left\{\left|\frac{k}{N} - \theta\right| \le \frac{1}{2}\right\} + \sigma^2_{NT} \log(\chi_k), \ \left|\frac{k}{N}\right| \le \frac{\tau}{2}.$$

Let $p_\zeta(x)$ be the probability density of $\zeta = \log(\chi)$. It is easy to check that Fisher's information of $p_\zeta(x)$ is

$$\int_{-\infty}^{\infty} \frac{[p'_\zeta(x)]^2}{p_\zeta(x)} \mathrm{d}x = 1.$$

Therefore with the standard technique we arrive at the equivalent Gaussian model

$$Z_k'' \overset{\mathbf{P}_\theta}{=} \mathbf{1}\left\{\left|\frac{k}{N}-\theta\right| \le \frac{1}{2}\right\} + \sigma_{NT}^2 \xi_k, \ \left|\frac{k}{N}\right| \le \frac{\tau_N}{2}.$$

This model is well approximated by continuous time model

$$\mathrm{d}Z^\tau(t) = \mathbf{1}\left\{\left|t-\theta\right| \le \frac{1}{2}\right\} + \frac{\sigma_{NT}^2}{\sqrt{N}}\mathrm{d}\widetilde{W}(t), \ |t| \le \frac{\tau_N}{2}.$$


Golubev, Yu. Nussbaum, M. Zhou, H. (2010) Asymptotic equivalence of spectral density estimation and Gaussian white noise. *Ann. Statist.* Vol. 38, No. 1, 181-214.

Finally, with τ_N-periodic function

$$\widetilde{I}_{\tau_N}(t) = \sum_{k=-\infty}^{\infty} \mathbf{1}\big\{|t - k\tau_N| \leq 0.5\big\}.$$

we arrive at the following continuous time equivalent model

$$\mathrm{d}Z^T(t) = \widetilde{I}_{\tau_N}(t-\theta) + \sigma_N^2 \sqrt{\frac{N\tau_N}{T_N}} \mathrm{d}\widetilde{W}(t), \ |t| \leq \frac{T_N}{2}.$$

The equivalence holds if

$$\lim_{N\to\infty} \frac{\sigma_N^2 N}{T_N/\tau_N} = \infty, \quad \lim_{N\to\infty} \frac{\sigma_N^2 \sqrt{N}}{T_N/\tau_N} = 0,$$

Doppler's shift estimation

Similarly, it can be shown if

$$\lim_{N\to\infty} \frac{\sigma_N^2 N}{T_N/\tau_N} = \infty, \quad \lim_{N\to\infty} \frac{\sigma_N^2 N^{2/3}}{T_N/\tau_N} = 0,$$

then the asymptotically equivalent statistical experiment for estimating θ, θ' is as follows

$$\mathrm{d}Z^T(t) = \widetilde{I}_{\tau_N}\big[(1-\theta')t - \theta\big]\mathrm{d}t + \sigma'_{NT}\mathrm{d}\widetilde{W}(t), \ |t| \le \frac{T_N}{2},$$

where

$$\sigma'_{NT} = \sigma_N^2\sqrt{\frac{N\tau_N}{T_N}}.$$

Statistical analysis of this model is simple since LLR is easily to compute

$$L(\theta', \theta; Z^T) \overset{\mathbf{P}}{=} -\frac{1}{2\sigma'^2_{NT}} \int_{-T_N/2}^{T_N/2} \Delta^2(t, \theta', \theta) \mathrm{d}t + \frac{1}{\sigma'_{NT}} \int_{-T_N/2}^{T_N/2} \Delta(t, \theta', \theta) \mathrm{d}\widetilde{W}(t),$$

where

$$\Delta(t, \theta', \theta) = \widetilde{I}_{\tau_N}\left[(1-\theta')t - \theta)\right] - \widetilde{I}_{\tau_N}\left[(1-\theta'_\circ)t - \theta_\circ)\right].$$

Let $\theta = 0$ and

$$r'_{NT} = \frac{2r_{NT}}{T_N} = \frac{\sigma_N^4 N \tau_N^2}{T_N^3}.$$

Then

$$L(\theta' + r'_{NT} t', 0; Z^T) - L(\theta', 0; Z^T) \overset{\mathbf{P}_{\theta'}}{=} -\frac{|t'|}{2} + \widetilde{W}(t'),$$

for all

$$|t'| \leq \frac{T_N^2}{\sigma_N^4 N \tau_N^2}.$$

Theorem

Assume

$$\lim_{N\to\infty} \frac{T_N^2}{\sigma_N^4 N \tau_N^2} = \infty.$$

Then

$$L\left(\theta' + r'_{NT} t', \theta + r_{NT} t; Z^T\right) - L\left(\theta', \theta; Z^T\right) \overset{\mathbf{P}_{\theta',\theta}}{\to}$$

$$\begin{cases} -\dfrac{|t'|}{2} + \widetilde{W}(t'), & |t'| \geq |t|, \\ -\dfrac{t'^2}{4t} - \dfrac{|t|}{4} + \widetilde{W}\left[\dfrac{(t'+t)^2}{4t}\right] + \widetilde{W}\left[-\dfrac{(t'-t)^2}{4t}\right], & |t'| < |t|. \end{cases}$$

Blind estimation

Unfortunately, the computational complexity of the estimate (θ', θ) is high, as it requires on the order of N^2 operations. Therefore, we will consider an approach that can reduce the complexity to $N \log(N)$ operations.
Consider the asymptotically equivalent statistical model for estimating θ'

$$\mathrm{d}Z^T(t) = \widetilde{I}_{\tau_N}\big(t - \theta - \theta' t\big)\mathrm{d}t + \sigma'_{NT}\mathrm{d}\widetilde{W}(t),\ |t| \le \frac{T_N}{2},$$

where

$$\widetilde{I}_{\tau_N}(t) = \sum_{k=-\infty}^{\infty} \mathbf{I}\big\{|t - k\tau_N| \le 0.5\big\},\ \sigma'_{NT} = \sigma_N^2\sqrt{\frac{N\tau_N}{T_N}},$$

and $\theta \in [-\tau_N/2, \tau_N/2]$ is a nuisance parameter.

The Fourier transform of $\widetilde{I}_{\tau_n}(\cdot)$ can easily computed

$$\widetilde{I}_{\tau_N}(t) = \frac{1}{\sqrt{\tau_N}} \sum_{k=-\infty}^{\infty} S_k \exp\left\{ \frac{2\pi \mathrm{i} k t}{\tau_N} \right\},$$

$$S_k = \frac{1}{\sqrt{\tau_N}} \mathrm{sinc}\left(\frac{k}{\tau_N} \right), \ \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

So,

$$\widetilde{I}_{\tau_N}(t - \theta - \theta' t) = \frac{1}{\sqrt{\tau_N}} \sum_{k=-\infty}^{\infty} S_k \exp\left\{ -\frac{2\pi k \theta}{\tau_N} \right\} \exp\left\{ \frac{2\pi \mathrm{i} k (1-\theta') t}{\tau_N} \right\}$$

and the idea in blind estimation is to consider

$$\mu_k = S_k \exp\left\{ -\frac{2\pi k \theta}{\tau} \right\}$$

as unknown nuisance parameters.

Thus, we estimate θ' based on

$$\mathrm{d}Z^T(t) = J\big(t, \theta', \mu\big)\mathrm{d}t + \sigma'_{NT}\mathrm{d}\widetilde{W}(t), \quad t \in [-T/2, T/2],$$

where

$$J(t, \theta', \mu) = \frac{1}{\sqrt{\tau_N}} \sum_{k=-\infty}^{\infty} \mu_k \exp\left\{ \frac{2\pi \mathrm{i} k(1-\theta')t}{\tau_N} \right\},$$

$$\sigma'_{NT} = \sigma_N^2 \sqrt{\frac{N\tau_N}{T_N}},$$

and $\mu_k \in \mathbb{C}$, $k = 0, \pm 1, \ldots$.

The LLR in the considered model is given by

$$L(\theta', \mu; Z^T) = -\frac{1}{\sigma'^2_{NT}} \int_{-T_N/2}^{T_N/2} |J(t, \theta', \mu)|^2 \, \mathrm{d}t + \frac{2}{\sigma'^2_{NT}} \mathrm{Re} \int_{-T_N/2}^{T_N/2} J^*(t, \theta', \mu) \, \mathrm{d}Z^T(t).$$

For given θ' the ML estimate of μ_k is easy to compute

$$\widehat{\mu}_k(Z^T; \theta') = \frac{1}{\sqrt{\tau_N}} \left(\frac{\tau_N}{T_N} \right) \int_{-T_N/2}^{T_N/2} \exp\left\{ -\frac{2\pi \mathrm{i} k (1 - \theta') t}{\tau_N} \right\} \mathrm{d}Z^T(t).$$

but the estimate

$$\widehat{\theta'} = \arg\max_{\theta'} L(\theta', \widehat{\mu}; Z^T) = \arg\max_{\theta'} \left\{ \sum_{k=-\infty}^{\infty} |\widehat{\mu}_k(Z^T; \theta')|^2 \right\}$$

is obviously not good.

This is why we consider the following estimate

$$\widehat{\theta}'_h = \arg\max_{\theta'} \left\{ \sum_{k=-\infty}^{\infty} h\left(\frac{k}{\tau_N}\right) \left|\widehat{\mu}_k(Z^T;\theta')\right|^2 \right\},$$

where $h(\cdot)$ is such that

- $h(-x) = h(x), x \in \mathbb{R}$,
- $0 \leq h(x) \leq 1,\ x \in \mathbb{R}$,
- $xh(x) \in \mathbb{L}_2$.

Theorem

Assume

$$\lim_{N\to\infty} \tau_N = \infty, \quad \lim_{N\to\infty} \frac{\sigma_N^2 N}{T_N/\tau_N} = \infty, \quad \lim_{N\to\infty} \frac{\sigma_N^4 N \tau_N^2}{T_N^2} = 0.$$

Then

$$\left(\widehat{\theta}_h - \theta'\right)/\widetilde{r}_{NT} \xrightarrow{\mathbf{P}_{\theta'}} \mathcal{K}(h)\xi, \ N \to \infty,$$

where $\xi \sim \mathcal{N}(0,1)$ *and*

$$\widetilde{r}_{NT} = r'_{NT} \times \sqrt{\tau_N},$$

$$\mathcal{K}(h) = \frac{\sqrt{3}}{2\pi} \left(\int_0^\infty x^2 h^2(x)\,\mathrm{d}x \right)^{1/2} \Big/ \int_0^\infty x^2 \mathrm{sinc}^2(x) h(x)\,\mathrm{d}x.$$

Notice

$$\arg\min_h \mathcal{K}(h) = \operatorname{sinc}^2(x),$$

$$\min_h \mathcal{K}(h) = \frac{\sqrt{3}}{2\pi}\left(\int_0^\infty x^2 \operatorname{sinc}^4(x)\,\mathrm{d}x\right)^{-1/2} \approx 1.7321.$$

HAPPY BIRTHDAY, YURY