Localization of radioactive source of unknown power in the space

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(Joint work with Y. Kutoyants and S. Dachian)

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Yury Kutoyants: A Personal & Scientific Journey

- First discovery: During my Ph.D., my tutor Prof. Cifarelli recommended Yury's book: *Parameter Estimation for Stochastic Processes* (1984) [>500 citations].
- First meeting (1995): We met in Padua, where Yury delivered a Ph.D. course on Inference for Stochastic Processes.
- Ph.D. in Le Mans (1995–1998): I completed my Ph.D. under Yury's supervision, focusing on nonparametric estimation for diffusion processes.
- Landmark milestone: Yury synthesized work in this field in his classic book *Statistical Inference for Ergodic Diffusion Processes* (2004) [>1200 citations].
- Today: Yury invited me to collaborate on Poisson process models—the beginning of a new scientific adventure!

Outline of the talk

1. Introduction & Literature Overview
   - Historical Background
   - Key Theoretical Contributions of Y. Kutoyants
2. Inference for Inhomogeneous Poisson Processes in 4D
   - Observation Model and signal
   - Asymptotic Properties of Estimators
     - MLE and BE
     - LSE
     - OSE

The general problem

- Problem of spatial source detection based on observations using multiple detectors.
- K detectors (sensors) $\mathbb{D}_1, \ldots, \mathbb{D}_K$ located at the points (x_k, y_k, z_k), $k = 1, 2, \ldots, K$ on the space, are given.
- The position of one source $\mathbb{S}$, $(x^*, y^*, z^*) = (\theta_1, \theta_2, \theta_3)$, is unknown.

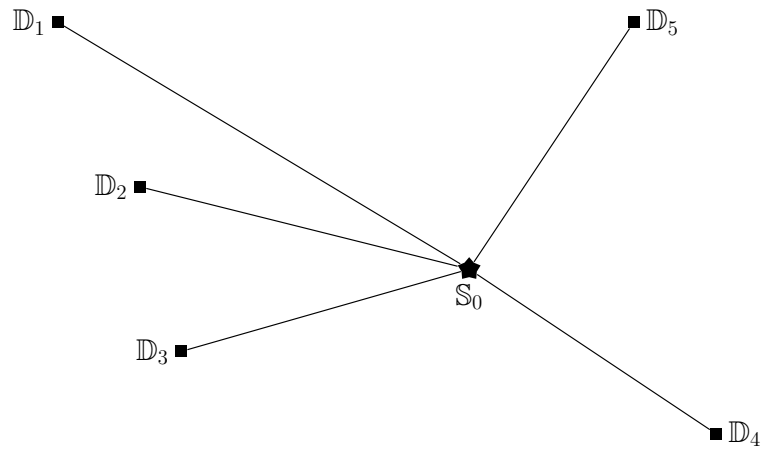


Figure: $\mathbb{S}_0$ is the position of the source and $\mathbb{D}_k$, $k = 1, \ldots, 5$ are positions of the sensors.

Origins

Practical & Military Origins

Position finding historically arose from the practical needs of navigators and surveyors, with wartime applications rapidly accelerating its development by military and defense requirements.

- 19th-Century Foundations (Bertot 1876, d'Ocagne 1896): Introduced early geometric solutions to estimate the most probable position from non-convergent lines.
- Statistical Formalization (Daniels 1951): bridged these classical geometric methods with modern estimation theory using least squares and maximum likelihood.

Note: *even for inherently 3D targets like submarines, classical approaches projected the problem onto a 2D plane* (x, y) *for simplicity.*

Engineering Approaches to Source Localization

Engineering Applications

These problems are of essential importance for practical applications and have been used in engineering for decades, though detailed statistical and mathematical studies have developed only recently.

- Chen et al. (2002) study maximum-likelihood source localization in sensor networks.
- Streit (2010) applies Poisson point processes to sensing, imaging, and tracking.
- Dardari et al. (2012) address satellite and terrestrial radio positioning techniques.
- Wang (2015) proposes linear least squares localization in sensor networks.
- Zekavat and Buehrer (2019) provide a comprehensive handbook on position location theory and practice.

Theoretical Foundations: Source Localization

Over the past decade, Yury Kutoyants and his research group have made fundamental contributions to signal source estimation. In this first part of my talk, I would like to review their main theoretical results.

Problem

Estimate the location of a source $\mathbb{S}$ emitting a Poisson signal.

- The true position of $\mathbb{S}$ is $\vartheta_0 \in \Theta \subset \mathbb{R}^2$, where Θ is convex and bounded.
- The source starts emitting at time $t = \tau_0$.
- $\mathbb{D}_j$ are K sensors receiving the signal.
- The received data are modeled as an inhomogeneous Poisson process $X^T = (X_1^T, \ldots, X_K^T)$, $X_j^T = (X_j(t), 0 \leq t \leq T)$.

The model

Intensity function of Poisson process

$$\lambda_j(\vartheta, t) = \lambda(t - \tau_j) + \lambda_0, \quad 0 \leq t \leq T, \quad j = 1, \ldots, K$$

- λ is the signal from the source, with $\lambda(t) = 0$ for $t \leq 0$;
- λ_0 is the intensity of the Poisson noise;
- $\tau_j = \tau_j(\vartheta) = \tau_0 + \frac{\|D_j - S_0\|}{\nu}$, is the arrival time of signal at j-th detector; τ_0 the starting time of emission, ν propagation rate;
- $S_0 = (x, y)$ is the unknown position of the source;
- $D_j = (x_j, y_j)$ are known positions of the K detectors;
- $\vartheta = (x, y) \in \Theta \subset \mathcal{R}^2$ or $\vartheta = (x, y, \tau) \in \Theta \subset \mathcal{R}^2 \times \mathcal{R}_+$;
- (x_0, y_0), the true position of the source.

Intensity function of Poisson process

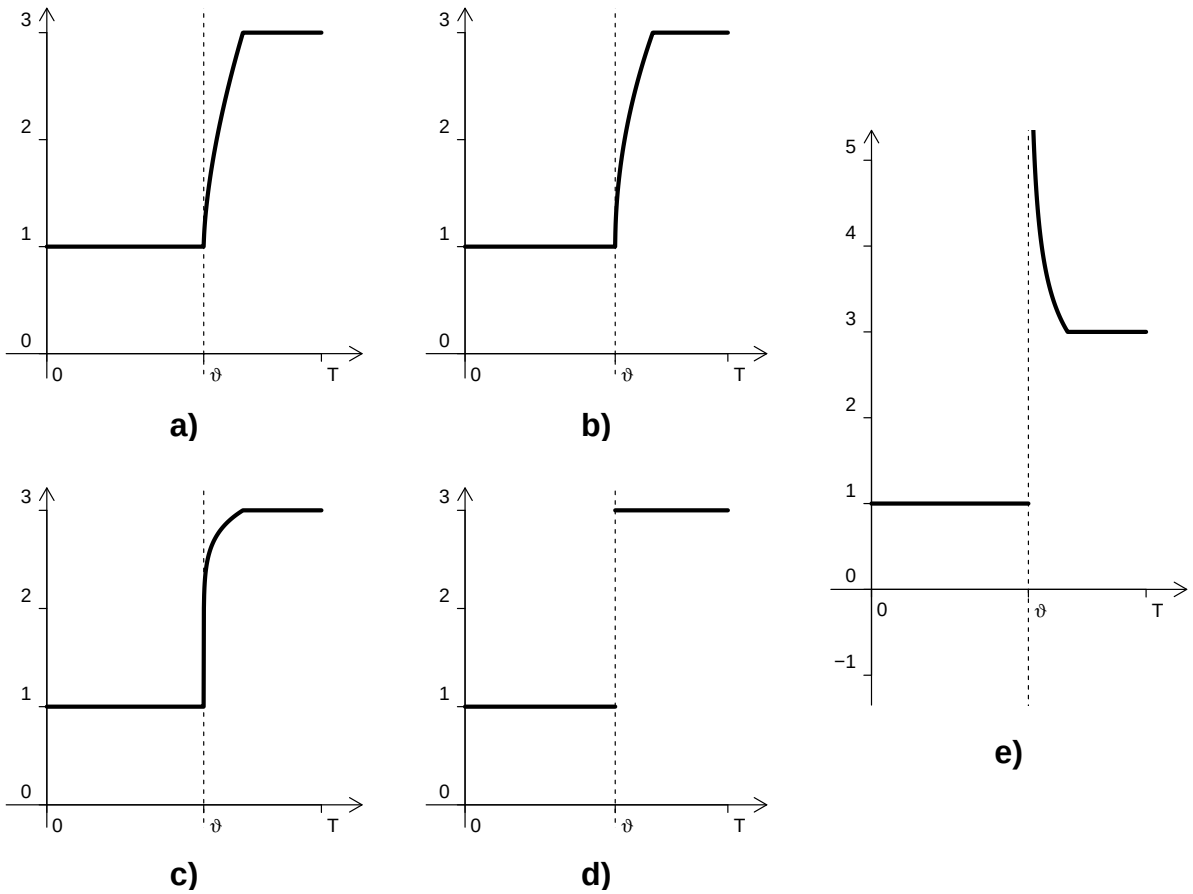


Figure: Intensity functions of different regularity: a) and b) smooth cases; c) cusp-type case; d) change-point case; e) non bounded intensity case.

Chernoyarov & Kutoyants (2020)

Large intensity model

The intensity has the form, $n \geq 1$

$$\lambda_{j,n}(\vartheta, t) = n\lambda(t - \tau_j(\vartheta)) + n\lambda_0, \quad 0 \leq t \leq T, \quad j = 1, \ldots, K$$

where the signal λ is a smooth function, $\tau_0 = 0$, λ_0 is known, ν is known.

Let $\vartheta_0 = (x_0, y_0)$ the true unknown position of the source.

Main Result - Smooth case

$$\sqrt{n}\left(\hat{\vartheta}_n - \vartheta_0\right) \Longrightarrow \zeta\left(\vartheta_0\right); \quad \sqrt{n}\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \zeta\left(\vartheta_0\right)$$

with $\zeta\left(\vartheta_0\right) \sim \mathcal{N}\left(0, \mathbf{I}\left(\vartheta_0\right)^{-1}\right)$. Moreover the convergence of polynomial moments holds and both estimators are asymptotically efficient.

Chernoyarov, Dachian & Kutoyants (2020)

Large intensity model with cusp

The signal λ has a cusp in 0:

$$\lambda(t) = a|t|^{\kappa}\mathbb{1}_{\{t\geq 0\}} + o(t), \quad \kappa \in (0, 1/2).$$

Here $\tau_0 = 0$ is known, λ_0, $a > 0$ known.

Let $\vartheta_0 = (x_0, y_0)$ the true unknown position of the source.

Main Result - Cusp case

$$n^{\frac{1}{2\kappa+1}}\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \zeta_2,$$

where ζ_1 is a random vector function of the fBm. Moreover the convergence of polynomial moments holds, and the BEs are asymptotically efficient.

Farinetto, Kutoyants & Top (2020)

Large intensity model with change point

The signal λ has a change-point (discontinuity) in 0:

$$\lambda(t) = a\mathbb{1}_{\{t\geq 0\}} + o(t).$$

Here $\tau_0 = 0$ is known, λ_0, $a > 0$ known.

Let $\vartheta_0 = (x_0, y_0)$ the true unknown position of the source.

Main Result - Change point case

$$n\big(\tilde{\vartheta}_n - \vartheta_0\big) \Longrightarrow \zeta_2,$$

where ζ_2 is a random vector function of two homogeneous Poisson processes. Moreover the convergence of polynomial moments holds, and the BEs are asymptotically efficient.

Arakelyan and Kutoyants (2019)

τ_0 is unknown and requiring estimation.

Observed Intensity Function:

$$\lambda_{j,n}(\vartheta, t) = n\lambda(t - \tau_j(\vartheta)) + n\lambda_0, \quad 0 \le t \le T, \quad j = 1, \dots, K$$

where the function λ is smooth.

$$\tau_j(\vartheta) = \tau + \tau_j(x, y), \quad j = 1, \dots, K$$

with $\vartheta = (x, y, \tau)^\top \in \Theta \subset \mathcal{R}^2 \times \mathcal{R}_+$, (x_0, y_0) position of $\mathbb{S}_0$

- First τ_j is estimated then τ_0 and (x_0, y_0)
- LSE is consistent: $\theta_n^* \to \vartheta_0$, where (x_0, y_0, τ_0) is the true value.

Chernoyarov, Dachian, Farinetto, & Kutoyants (2022)

τ_0 is unknown and requiring estimation.

Gaussian Model

$$\mathrm{d}X_k(t) = S_k(t-\tau_{0,k})\mathrm{d}t+\varepsilon\sigma_k(t)\mathrm{d}W_k(t), \quad X_k(0) = 0, \quad 0 \leq t \leq T$$

- S smooth function
- MLE an BE's are consistent, asymptotically Normal, convergence of polynomial moments, and asympt. efficient.
- LSE is asymptotically Normal

Kutoyants (2023, Chapter 5)

Provides a comprehensive review of all preceding results.
Moreover, for the case $\vartheta = (x, y, \tau)$, he proves that the asymptotic properties of MLE, BE, and LSE extend fully to the Poisson model.

Chernoyarov, Dachian, Farinetto, & Kutoyants (2024)

Two sources. $\tau_0 = 0$

Observed Intensity Function:

$$\lambda_{j,n}(\vartheta, t) = nS_{1,j}(\vartheta^{(1)}, t) + nS_{2,j}(\vartheta^{(2)}, t) + n\lambda_0, \qquad 0 \le t \le T,$$

with $S_{i,j}(\vartheta^{(i)}, t) = \psi\big(t - \tau_j(\vartheta^{(i)})\big) S_{i,j}(t)$.

- Regular case: MLE and BE are consistent, asymptotically normal, have convergent polynomial moments, and are asymptotically efficient.
- Cusp case: BE is consistent, $n^{\frac{1}{2\kappa+1}}\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \xi$, has convergent polynomial moments, and is asymptotically efficient.
- Change point case: BE is consistent, $n\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \eta$, has convergent polynomial moments, and is asymptotically efficient.

Localization of a Radioactive Source

- The problem of detecting and localizing a radioactive source plays a central role in this field.
- The observation model is an inhomogeneous Poisson process, where detection reduces to a parameter estimation problem across various settings.
- Wan *et al.* (2012) describe in detail the relationship between the physical properties of the source and the corresponding Poisson process model.
- Similar problems were studied by Brennan *et al.* (2004), Baidoo-Williams *et al.* (2015), and in the monograph by Knoll (2010).

The Observation Model

- K detectors $D_1, \ldots, D_K$ located at known spatial positions $(x_k, y_k, z_k) \in \mathcal{R}^3$, for $k = 1, \ldots, K$
- One source S located at an unknown position (x^*, y^*, z^*)
- $X^T = (X_1^T, \ldots, X_K^T)$, where $X_j^T = (X_j(t) : 0 \leq t \leq T)$

Intensity function

$$\lambda_k(\vartheta, t) = \frac{\theta_1 s_k(t)}{\rho_k(\vartheta)^2} + \lambda_0, \qquad 0 \leq t \leq T$$

- $\vartheta = (\theta_1, \theta_2, \theta_3, \theta_4)^\top \in \Theta \subset \mathcal{R}^4$, with $(x^*, y^*, z^*) = (\theta_2, \theta_3, \theta_4)$;
- $\theta_1 > 0$ is the unknown source amplitude;
- $\rho_k(\vartheta) = \left[(x_k - \theta_2)^2 + (y_k - \theta_3)^2 + (z_k - \theta_4)^2\right]^{1/2}$;
- $s_k(t) \geq 0$ is a known signal modulation function;
- $\lambda_0 > 0$ is the background Poisson noise intensity.

Parameter Space

- The parameter space $\Theta \subset \mathbb{R}^4$ is assumed to be open and bounded.
- The signal amplitude satisfies $\theta_1 \in (\alpha, \beta)$, where $0 < \alpha < \beta < \infty$.
- To avoid singularities at detector locations, the source cannot lie within an ε-neighborhood ($\varepsilon > 0$) of any detector.
- This technical condition restricts the admissible spatial set $\Theta^{(3)}$ for the source position $(\theta_2, \theta_3, \theta_4)^\top$.

The likelihood

- $\left\{\mathbf{P}_{\vartheta}^{T}, \vartheta \in \Theta\right\}$ measures on $\mathcal{D}[0, T]$ are equivalent
- $\mathbf{P}_{\lambda_0}^{T}$ the measure of the homogeneous Poisson process of intensity λ_0 (noise)

The likelihood ratio formula

$$L(\vartheta, X^T) = \frac{\mathrm{d}\mathbf{P}_{\vartheta}^{T}}{\mathrm{d}\mathbf{P}_{\lambda_0}^{T}} = \exp\left(\sum_{k=1}^{K}\int_0^T \ln\frac{\lambda_k(\vartheta, t)}{\lambda_0}\,\mathrm{d}X_k(t) - \sum_{k=1}^{K}\int_0^T [\lambda_k(\vartheta, t) - \lambda_0]\,\mathrm{d}t\right)$$
$$= \exp\left(\sum_{k=1}^{K}\int_0^T \ln\left[\frac{\theta_1 s_k(t)}{\rho_k(\vartheta)^2\lambda_0}+1\right]\mathrm{d}X_k(t) - \sum_{k=1}^{K}\frac{\theta_1 S_k(T)}{\rho_k(\vartheta)^2}\right), \qquad \vartheta \in \Theta,$$

where we denoted

$$S_k(T) = \int_0^T s_k(t)\,\mathrm{d}t \qquad (*)$$

Maximum Likelihood and Bayesian Estimators

Definition

The maximum likelihood estimator (MLE) $\hat{\vartheta}_T$ is

$$L(\hat{\vartheta}_T, X^T) = \sup_{\vartheta \in \Theta} L(\vartheta, X^T).$$

Definition

The Bayesian estimator (BE) is the conditional expectation

$$\tilde{\vartheta}_T = \int_\Theta \vartheta \, p\left(\vartheta | X^T\right) \, \mathrm{d}\vartheta = \frac{\int_\Theta \vartheta \, p(\vartheta) \; L(\vartheta, X^T) \, \mathrm{d}\vartheta}{\int_\Theta p(\vartheta) \; L(\vartheta, X^T) \, \mathrm{d}\vartheta},$$

where $p(\vartheta)$, $\vartheta \in \Theta$ is the *a priori* density, a known, continuous, positive function.

The Fisher Information

The Fisher information matrix

$$\mathbf{I}_T(\vartheta) = \mathbf{E}_\vartheta \left[\left(\frac{\partial \ln L(\vartheta, X^T)}{\partial \vartheta} \right) \left(\frac{\partial \ln L(\vartheta, X^T)}{\partial \vartheta} \right)^\top \right]$$
$$= \sum_{k=1}^{K} \int_0^T \frac{1}{\lambda_k(\vartheta, t)} \left(\frac{\partial \lambda_k(\vartheta, t)}{\partial \vartheta} \right) \left(\frac{\partial \lambda_k(\vartheta, t)}{\partial \vartheta} \right)^\top \mathrm{d}t$$

is a 4×4 matrix.

Periodic Signal

Likelihood and Fisher Information

For $T = n\tau + r$ with $0 \leq r < \tau$, (*) became
$S_k(T) = nS_k(\tau)\left(1 + O\left(n^{-1}\right)\right)$, and setting $S_k(\tau) = \int_0^\tau s_k(t)\,\mathrm{d}t$

$$\ln L(\vartheta, X^n) = \sum_{k=1}^{K} \int_0^{n\tau} \ln\left[1 + \frac{\theta_1 s_k(t)}{\rho_k(\vartheta)^2 \lambda_0}\right] \mathrm{d}X_k(t)$$
$$- n \sum_{k=1}^{K} \frac{\theta_1 S_k(\tau)}{\rho_k(\vartheta)^2} + O_P(1)$$

The Fisher Information is $\mathbf{I}_n(\vartheta) = n\mathbf{I}(\vartheta) + O(1)$, where

$$\mathbf{I}(\vartheta) = \sum_{k=1}^{K} \int_0^\tau \frac{1}{\lambda_k(\vartheta, t)} \left(\frac{\partial \lambda_k(\vartheta, t)}{\partial \vartheta}\right) \left(\frac{\partial \lambda_k(\vartheta, t)}{\partial \vartheta}\right)^\top \mathrm{d}t$$

Assumptions

Condition (1)

$$G(\nu, \vartheta_0) = \inf_{\|\vartheta - \vartheta_0\| \geq \nu} \sum_{k=1}^{K} \left| \frac{\theta_1}{\rho_k(\vartheta)^2} - \frac{\theta_{0,1}}{\rho_k(\vartheta_0)^2} \right| > 0$$

for any $\nu > 0$. Here $\vartheta_0 = (\theta_{0,1}, \theta_{0,2}, \theta_{0,3}, \theta_{0,4})$.

Condition (2)

We suppose that the Fisher information matrix is uniformly non-degenerate:

$$\inf_{\vartheta \in \Theta}, \inf_{\|\mathbf{e}\|=1} \mathbf{e}^\top \mathbf{I}(\vartheta)\, \mathbf{e} > 0,$$

where $\mathbf{e} \in \mathbb{R}^4$ is a unit vector representing a direction in the parameter space.

Asymptotic properties of MLE and BE

Main Result for MLE and BE

Let the regularity conditions (1) and (2) hold. Then the MLE and BE are, uniformly on compacts, consistent, asymptotically normal

$$\sqrt{n}\left(\hat{\vartheta}_n - \vartheta_0\right) \Longrightarrow \zeta\left(\vartheta_0\right), \qquad \sqrt{n}\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \zeta\left(\vartheta_0\right),$$

where $\zeta(\vartheta_0) \sim \mathcal{N}\left(0, \mathbf{I}\left(\vartheta_0\right)^{-1}\right)$. Moreover, for any $p > 0$

$$n^{p/2}\mathbf{E}_{\vartheta_0}\|\hat{\vartheta}_T - \vartheta_0\|^p \to \mathbf{E}_{\vartheta_0}\left\|\zeta\left(\vartheta_0\right)\right\|^p,$$
$$n^{p/2}\mathbf{E}_{\vartheta_0}\|\tilde{\vartheta}_T - \vartheta_0\|^p \to \mathbf{E}_{\vartheta_0}\left\|\zeta\left(\vartheta_0\right)\right\|^p.$$

Both estimators are asymptotically efficient.

Asymptotic properties of MLE and BE

- The proof is based on Ibragimov and Khasminskii (1981, Theorems 3.1.1 and 3.2.1).
- The argument follows the structure of Theorem 2.2 in Kutoyants (2023).
- Conditions (1) and (2) assumed here guarantee that the hypotheses of Theorem 2.2 are satisfied.
- A crucial role is played by the weak convergence of the likelihood ratio process.

Likelihood ratio

Let for $u \in \mathbb{U}_n = \left(u \ : \ \vartheta_0 + \varphi_n u \in \Theta\right)$

$$Z_n\left(\vartheta_0, u\right) = \frac{L\left(\vartheta_0 + \varphi_n u, X^n\right)}{L\left(\vartheta_0, X^n\right)},$$

Asymptotic properties of MLE and BE

Lemma 1

The random function $Z_n(\vartheta_0, u), u \in \mathbb{U}_n$ admits the representation

$$Z_n(\vartheta_0, u) = \exp\left(u^\top \Delta_n(\vartheta_0, X^n) - \frac{1}{2} u^\top \mathbf{I}(\vartheta_0)\, u + r_n\right),$$

where $\Delta_n(\vartheta_0, X^n) \Longrightarrow \mathcal{N}(0, \mathbf{I}(\vartheta_0)), \qquad r_n \longrightarrow 0.$

Lemma 2

There exists a constant $C > 0$ such that for $\|u_1\| + \|u_2\| \le R$,

$$\mathbf{E}_{\vartheta_0} \left| Z_n(\vartheta_0, u_2)^{1/6} - Z_n(\vartheta_0, u_1)^{1/6} \right|^6 \le C \left(1 + R^6\right) \|u_2 - u_1\|^6.$$

Lemma 3

There exists a constant $\kappa > 0$ such that

$$\mathbf{P}_{\vartheta_0}\left(Z_n(\vartheta_0, u) > e^{-c\|u\|^2}\right) \le e^{-\kappa\|u\|^2}$$

Least square estimator

Remark

The MLE has no closed-form expression even when the source position is known:

$$\sum_{k=1}^{K} \int_0^T \left[\hat{\theta}_{1,T} + \frac{\rho_k^2 \lambda_0}{s_k(t)} \right]^{-1} \mathrm{d}X_k(t) = \sum_{k=1}^{K} \frac{S_k(T)}{\rho_k^2}.$$

Observing that

$$\mathbf{E}_\vartheta X_k(T) = \frac{\theta_1 S_k(T)}{\rho_k(\vartheta)^2} + \lambda_0 T, \text{ and } \quad \frac{X_k(T) - \lambda_0 T}{S_k(T)} \longrightarrow \frac{\theta_1}{\rho_k(\vartheta)^2}$$

We have

$$M_{k,T} = \frac{S_k(T)}{X_k(T) - \lambda_0 T} \longrightarrow \frac{\rho_k(\vartheta_0)^2}{\theta_{0,1}} \equiv M_k(\vartheta_0),$$

Least square estimator

The LSE can be defined as

LSE

Let $\boldsymbol{\rho}(\vartheta)^2 = (\rho_1(\vartheta)^2, \ldots, \rho_K(\vartheta)^2)^\top$, the LSE is defined as

$$\breve{\vartheta}_T = \arg\inf_{\vartheta\in\Theta} \left\| \theta_1 \mathbf{M}_T - \boldsymbol{\rho}(\vartheta)^2 \right\|_K .$$

where $\mathbf{M}_T = (M_{1,T}, \ldots, M_{K,T})^\top$.

Identifiability condition (3)

$$g(\vartheta_0, \nu) \equiv \inf_{\|\vartheta-\vartheta_0\|\geq\nu} \left\| \boldsymbol{\rho}(\vartheta)^2 - \frac{\theta_1}{\theta_{0,1}} \boldsymbol{\rho}(\vartheta_0)^2 \right\|_K > 0, \text{for any } \nu > 0.$$

Asymptotic properties of LSE

Proposition

Let the identifiability condition (3) hold. Then for any integer $p \geq 1$ there exists a constant $C = C(p) > 0$ such that for any $\nu > 0$

$$\mathbf{P}_{\vartheta_0}\left(\|\breve{\vartheta}_T - \vartheta_0\| > \nu\right) \leq \frac{C}{T^p g(\vartheta_0, \nu)^{2p}} + \frac{C}{T^p}.$$

Proposition

Let the identifiability condition (3) hold. Let the matrix $\mathbb{A}(\vartheta)$ be uniformly nondegenerate, $\inf_{\vartheta \in \Theta} \mathrm{Det}\, \mathbb{A}(\vartheta) \neq 0$. Then the LSE $\breve{\vartheta}_T = \mathbb{A}_T^{-1} b_T$ and is asymptotically normal.

$$\sqrt{T}\left(\breve{\vartheta}_T - \vartheta_0\right) \Longrightarrow \mathcal{N}(0, \mathbb{C}(\vartheta_0))$$

One Step Estimator

- The MLE is asymptotically efficient but its computation is numerically expensive
- The LSE is obtained as the solution of a linear system but the error of estimation is greater
- The One-step MLE was introduced by Fisher (1925) and studied by Le Cam (1956) and (1960)
- Kutoyants idea is based on a 2-step estimation procedure:
  1. a consistent estimator is defined on a learning interval,
  2. a One-step MLE-process is defined on the rest of the observation period with the help of the score function

One Step Estimator

1. the LSE $\breve{\vartheta}_{T_\delta}$ by observations $X^{T_\delta} = (X_k(t), 0 \le t \le T_\delta)$, $k = 1, \dots, K$, where $T_\delta = T^\delta$, $\delta \in (1/2, 1)$.
2. One-step estimator takes the form

$$\vartheta_T^\star = \breve{\vartheta}_{T_\delta} + \mathbf{I}_{T_\delta}(\breve{\vartheta}_{T_\delta})^{-1} \sum_{k=1}^{K} \int_{T_\delta}^{T} \frac{\dot{\lambda}_k(\breve{\vartheta}_{T_\delta}, s)}{\lambda_k(\breve{\vartheta}_{T_\delta}, s)} \left[\mathrm{d}X_k(s) - \lambda_k(\breve{\vartheta}_{T_\delta}, s)\, \mathrm{d}s \right]$$

where the Fisher information matrix is

$$\mathbf{I}_{T_\delta}(\vartheta) = \sum_{k=1}^{K} \int_{T_\delta}^{T} \frac{\dot{\lambda}_k(\vartheta, s)\dot{\lambda}_k(\vartheta, s)^\top}{\lambda_k(\vartheta, s)}\, \mathrm{d}s.$$

Theorem

Let the conditions (1)-(3) hold. Then the estimator $\vartheta_T^\star$ is consistent and asymptotically normal

$$\sqrt{T}\,(\vartheta_T^\star - \vartheta_0) \Longrightarrow \mathcal{N}\left(0, \mathbf{I}(\vartheta_0)^{-1}\right).$$

What else?

Thank you for your attention !!

&

Happy birthday Yury !!!

Thank you, Yury, for everything! (Remembering Bergamo, 2023)

