



Approximate likelihood inference for stochastic differential equations using splitting schemes

Based on joint work with Susanne Ditlevsen and Adeline Samson

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Likelihood inference for SDE models

$$dX_t = F(X_t;\theta)dt + \sigma(X_t;\theta)dW_t \qquad \theta \in \Theta \subseteq \mathbb{R}^p$$

State space $D \subseteq \mathbb{R}^d$

F d-dimensional, Σ $d \times d$ – *matrix*, W d-dimensional Wiener process

Data: $X_{t_0}, \ldots, X_{t_n}$, $\Delta_i = t_i - t_{i-1}$

Log-likelihood function: $$\ell_n(\theta) = \sum_{i=1}^{n} \log p_{\Delta_i}(X_{t_i} \mid X_{t_{i-1}};\theta)$$

$p_\Delta(\cdot \mid x;\theta)$ is the transition density, i.e. density of $X_\Delta \mid X_0 = x$

Approximations of $p_\Delta(\cdot \mid x;\theta)$ or $\partial_\theta \log p_\Delta(\cdot \mid x;\theta)$ are usually needed



Approximate likelihood function based on splitting schemes

Splitting schemes for simulation of SDEs: Bensoussan et al. (1992) , Ableidinger & Buckwar (2016), Buckwar et al. (2022) and others

Splitting schemes for inference: Buckwar, Samson & Tamborrino (2022), Pilipovic, Samson & Ditlevsen (2024), Fang, Spanò & Tamborrino (2026) and others

Pilipovic, Samson & Ditlevsen (2024): Approximation of $p_\Delta(\cdot \mid x; \theta)$ based on Lie-Trotter and Strang splitting scheme

$$dX_t = F(X_t; \theta)dt + \Sigma dW_t$$

Σ is a $d \times d$ matrix of full rank

Rewrite the SDE:

$$dX_t = -\Psi_\theta(X_t - \mu_\theta)dt + N(X_t; \theta)dt + \Sigma dW_t$$

where

$$N(x; \theta) = F(x; \theta) + \Psi_\theta(x - \mu_\theta)$$



Approximate likelihood function based on Lie-Trotter splitting

$$dX_t = -\Psi_\theta(X_t - \mu_\theta)dt + N(X_t;\theta)dt + \Sigma dW_t$$

Splitting of X: $dX_t^1 = -\Psi_\theta(X_t^1 - \mu_\theta)dt + \Sigma dW_t, \qquad dX_t^2 = N(X_t^2;\theta)dt$

$$X_{t+\Delta}^1 = m_\Delta(X_t^1;\theta) + \xi_\Delta, \qquad X_{t+\Delta}^2 = f_{\Delta,\theta}(X_t^2)$$

where

$$m_\Delta(x;\theta) = e^{-\Psi_\theta\Delta}x + (I_d - e^{-\Psi_\theta\Delta})\mu_\theta$$

$$\xi_\Delta \sim N_d(0,\Omega_\Delta(\theta,\Sigma)), \qquad \Omega_\Delta(\theta,\Sigma) = \int_0^\Delta e^{-\Psi_\theta(\Delta-u)}\Sigma\Sigma^\top e^{-\Psi_\theta^\top(\Delta-u)}du$$



Approximate likelihood function based on Lie-Trotter splitting

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Lie-Trotter scheme: $X_{t+\Delta}^{LT} = m_\Delta(f_{\Delta,\theta}(X_t^{LT});\theta)) + \xi_\Delta$



Approximate likelihood function based on Lie-Trotter splitting

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Lie-Trotter scheme: $X_{t+\Delta}^{LT} = m_\Delta(f_{\Delta,\theta}(X_t^{LT});\theta)) + \xi_\Delta$

Lie-Trotter splitting approximate log-likelihood:

$$\ell_n^{LT}(\theta,\Sigma) = -\tfrac{n}{2}\log\det\Omega_{\Delta_i}(\theta,\Sigma)$$
$$-\tfrac{1}{2}\sum_{i=1}^n (X_{t_i} - m_{\Delta_i}(f_{\Delta_i,\theta}(X_{t_{i-1}});\theta))^\top \Omega_{\Delta_i}(\theta,\Sigma)^{-1}(X_{t_i} - m_{\Delta_i}(f_{\Delta_i,\theta}(X_{t_{i-1}});\theta))$$



Approximate likelihood function based on Strang splitting

$$dX_t = -\Psi_\theta(X_t - \mu_\theta)dt + N(X_t;\theta)dt + \Sigma dW_t$$

$$dX_t^1 = -\Psi_\theta(X_t^1 - \mu_\theta)dt + \Sigma dW_t, \qquad dX_t^2 = N(X_t^2;\theta)dt$$

$$X_{t+\Delta}^1 = m_\Delta(X_t^1;\theta) + \xi_\Delta, \qquad X_{t+\Delta}^2 = f_{\Delta,\theta}(X_t^2)$$

Strang scheme: $X_{t+\Delta}^S = f_{\Delta/2,\theta}\left(m_\Delta(f_{\Delta/2,\theta}(X_t^S);\theta)) + \xi_\Delta\right)$

Strang splitting approximate log-likelihood:

$$\ell_n^S(\theta,\Sigma) =$$

$$-\tfrac{n}{2}\log\det\Omega_{\Delta_i}(\theta,\Sigma) - \tfrac{1}{2}\sum_{i=1}^n Z_i(\theta)^\top \Omega_{\Delta_i}(\theta,\Sigma)^{-1} Z_i(\theta) + \sum_{i=1}^n \log|\det D\, f_{\Delta_i/2,\theta}(X_{t_i})|$$

where $Z_i(\theta) = f_{\Delta_i/2,\theta}^{-1}(X_{t_i}) - m_{\Delta_i}(f_{\Delta_i/2,\theta}(X_{t_{i-1}});\theta)$



State dependent diffusion coefficient

$$dX_t = F(X_t;\theta)dt + \Sigma(X_t;\theta)dW_t$$

$$dX_t^1 = b(X_t^1;\theta)dt + \Sigma(X_t^1;\theta)dW_t, \qquad dX_t^2 = N(X_t^2;\theta)dt$$

where

$$N(x;\theta) = F(x;\theta) - b(x;\theta)$$



State dependent diffusion coefficient

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where

$$N(x;\theta) = F(x;\theta) - b(x;\theta)$$

Generator of X^1:

$$L_\theta f(x) = \sum_{i=1}^{d} b_i(x,\theta)\partial_{x_i} f(x) + \frac{1}{2}\sum_{i,j=1}^{d} V_{ij}(x,\theta)\partial^2_{x_i,x_j} f(x),$$

where $f : D \mapsto \mathbb{R}$ is twice differentiable and $V(x;\theta) = \Sigma(x;\theta)\Sigma(x;\theta)^\top$

b is chosen such that we can find explicit eigenfunctions and eigenvalues

$$\phi_{j,\theta},\ \lambda_{j,\theta} > 0,\ j = 1,\ldots,m,$$

of the generator of X^1. Specifically,

$$L_\theta \phi_{j,\theta} = -\lambda_{j,\theta}\phi_{j,\theta}$$



Martingale estimating functions for X^1

Kessler & Sørensen (1999) ($d = 1$):

Under weak regularity conditions

$$G_n(\theta) = \sum_{i=1}^{n} g(t_i - t_{i-1}, X^1_{t_i}, X^1_{t_{i-1}}, \theta),$$

with

$$g(\Delta, y, x, \theta) = \sum_{j=1}^{m} a_{j,\theta}(\Delta, x) \left[\phi_{j,\theta}(y) - e^{-\lambda_{j,\theta}\Delta}\phi_{j,\theta}(x)\right]$$

is a martingale estimating function for X^1, i.e. for all $\Delta > 0$, $x \in D$, $\theta \in \Theta$

$$E_\theta(g(\Delta, X^1_\Delta, x, \theta) \,|\, X^1_0 = x) = 0$$

$\hat{\theta}_n$ is called a G_n-estimator if $P(G_n(\hat{\theta}_n) = 0) \to 1$ as $n \to \infty$



Martingale estimating functions for X^1

The martingale property of

$$G_n(\theta) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{j,\theta}(t_i - t_{i-1}, X^1_{t_{i-1}}) \left[\phi_{j,\theta}(X^1_{t_i}) - e^{-\lambda_{j,\theta}(t_i - t_{i-1})} \phi_{j,\theta}(X^1_{t_{i-1}}) \right]$$

holds for multivariate diffusions under the condition

$$\int_0^t E_\theta \left(\nabla \phi_{j,\theta}(X_s^{[1]})^\top V(X_s^{[1]}) \nabla \phi_{j,\theta}(X_s^{[1]}) \right) ds < \infty \quad \text{for all } j, \theta \text{ and } t > 0$$

$\nabla \phi_{j,\theta}$ is the gradient of $\phi_{j,\theta}$ and $V(x;\theta) = \Sigma(x;\theta)\Sigma(x;\theta)^\top$



Approximate score function for X based on LT-splitting

Ditlevsen, Samson & Sørensen (2026)

$$dX_t^1 = b(X_t^1;\theta)dt + \Sigma(X_t^1;\theta)dW_t$$

$$X_{t+\Delta}^1 = \varphi_{\Delta,\theta}(X_t^1) \qquad X_{t+\Delta}^2 = f_{\Delta,\theta}(X_t^2)$$

Martingale estimating function/approximate score for X^1 $\quad (t_i = i\Delta)$

$$G_n(\theta) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{j,\theta}(\Delta, X_{t_{i-1}}^1)\left[\phi_{j,\theta}(X_{t_i}^1) - e^{-\lambda_{j,\theta}\Delta}\phi_{j,\theta}(X_{t_{i-1}}^1)\right]$$

Lie-Trotter scheme: $\quad X_{t+\Delta}^{LT} = \varphi_{\Delta,\theta} \circ f_{\Delta,\theta}(X_t^{LT})$

Approximate score function for X based on Lie-Trotter splitting

$$G_n^{LT}(\theta) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{j,\theta}(\Delta, X_{t_{i-1}}^{LT})\left[\phi_{j,\theta}(X_{t_i}^{LT}) - e^{-\lambda_{j,\theta}\Delta}\phi_{j,\theta}(f_{\Delta,\theta}(X_{t_{i-1}}^{LT}))\right]$$



Approximate score function for X based on Strang splitting

Ditlevsen, Samson & Sørensen (2026)

$$dX_t^1 = b(X_t^1;\theta)dt + \Sigma(X_t^1;\theta)dW_t$$

$$X_{t+\Delta}^1 = \varphi_{\Delta,\theta}(X_t^1) \qquad X_{t+\Delta}^2 = f_{\Delta,\theta}(X_t^2)$$

Strang scheme: $X_{t+\Delta}^S = f_{\Delta/2,\theta} \circ \varphi_{\Delta,\theta} \circ f_{\Delta/2,\theta}(X_t^S)$

Martingale estimating function/approximate score for $Y_t = f_{\Delta/2,\theta}(X_t^1)$:

$$G_n(\theta) = \sum_{i=1}^n \sum_{j=1}^m a_{j,\theta}(\Delta, Y_{t_{i-1}}) \left[\phi_{j,\theta}(f_{-\Delta/2,\theta}(Y_{t_i})) - e^{-\lambda_{j,\theta}\Delta}\phi_{j,\theta}(f_{-\Delta/2,\theta}(Y_{t_{i-1}}))\right]$$

$f_{\Delta/2,\theta}^{-1}(x) = f_{-\Delta/2,\theta}(x)$

Approximate score function for X based on Strang splitting:

$$G_n^S(\theta) = \sum_{i=1}^n \sum_{j=1}^m a_{j,\theta}(\Delta, X_{t_{i-1}}^S) \left[\phi_{j,\theta}(f_{-\Delta/2,\theta}(X_{t_i}^S)) - e^{-\lambda_{j,\theta}\Delta}\phi_{j,\theta}(f_{\Delta/2,\theta}(X_{t_{i-1}}^S))\right]$$



Approximate martingale estimating function

$$G_n(\theta) = \sum_{i=1}^{n} g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta)$$

is called an approximate martingale function of order $\kappa \geq 2$ if

$$E_\theta\left(g(\Delta, X_\Delta, x; \theta) \,|\, X_0 = x\right) = O(\Delta^\kappa) \quad \text{for all } \Delta > 0, x \in D, \theta \in \Theta$$

Theorem

G_n^{LT} is an approximate martingale estimating function of order 2

G_n^{S} is an approximate martingale estimating function of order 3

Follows from Lemma 2.4 in Sørensen (2024)



Approximate martingale estimating functions

Lemma 2.4 in Sørensen (2024)

Under smoothness and polynomial growth conditions on the functions g, F and Σ,

$$G_n(\theta) = \sum_{i=1}^{n} g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta)$$

is an approximate martingale estimating function of order $\kappa \geq 2$ if and only if

$$\sum_{j=0}^{k} \binom{k}{j} L_\theta^{k-j}(g_i^{(j)}(\theta))(x,x) = 0, \quad k = 0, \ldots, \kappa - 1, i = 1, \ldots, p$$

for all $x \in D$ and $\theta \in \Theta$

$$L_\theta(g_i^{(j)}(\theta))(y,x) = \sum_{k=1}^{d} F_k(y;\theta)\partial_{y_k} g_i^{(j)}(y,x;\theta) + \frac{1}{2}\sum_{k,\ell=1}^{d} V_{k,\ell}(y;\theta)\partial^2_{y_k,y_\ell} g_i^{(j)}(y,x;\theta)$$

$$g^{(j)}(y,x;\theta) = \partial_\Delta^j g(0,y,x;\theta)$$



Pearson diffusions

$$dX_t = -\psi_\theta(x - \mu_\theta)dt + \Sigma(X_t)dW_t, \quad V(x) = \Sigma(x)\Sigma(x)^\top$$

$V_{ij}(x) = \sum_{|\boldsymbol{m}|\leq 2} \gamma_{\boldsymbol{m}}^{(ij)} x^{\boldsymbol{m}}, \quad i,j = 1,\ldots,d, \quad \gamma_{\boldsymbol{m}}^{(ij)} = \gamma_{\boldsymbol{m}}^{(ji)}$

$\boldsymbol{m} = (m_1,\ldots,k_d) \in \mathbb{N}_0^d$, $|\boldsymbol{m}| = k_1 + \cdots + k_d$, $c_{\boldsymbol{k}} \in \mathbb{R}$ and $x^{\boldsymbol{k}} = x_1^{k_1} \cdots x_d^{k_d}$

The generator maps polynomials of order n into polynomials of order n

$d = 1$ Wong(1964), Forman & Sørensen (2008)

Ornstein-Uhlenbeck process
skew t process (Pearson's type IV)
square root process (CIR)
inverse gamma diffusion (mean-reverting GBM, GARCH-diffusion)
F diffusion
Jacoby diffusion

Generalizations to the multivariate case:

Cuchiero, Keller-Ressel & Teichmann (2012) (polynomial processes)
Filipović & Larsson (2016) (polynomial diffusions)

Transformations of Pearson diffusions



High frequency asymptotics: optimal rate, Fisher information

We assume that $\theta = (\alpha, \beta) \in \Theta_1 \times \Theta_2$, $\Theta_i \subseteq \mathbb{R}^{p_i}$ $(p = p_1 + p_2)$, and that

$$dX_t = F(X_t; \alpha)dt + \Sigma(X_t; \beta)dW_t$$

is ergodic with invariant density ν_θ.

We consider the high frequency/infinite horizon asymptotic scenario

$$n \to \infty, \qquad \Delta_n \to 0, \qquad n\Delta_n \to \infty$$

Gobet (2002): The model is LAN. Under $P_{\alpha,\beta}$ optimal rate of convergence is

$$\hat{\alpha}_n - \alpha = O_P\left(\frac{1}{\sqrt{n\Delta}}\right)$$

$$\hat{\beta}_n - \beta = O_P\left(\frac{1}{\sqrt{n}}\right)$$



High frequency asymptotics: rate optimality and efficiency

High frequency/infinite horizon asymptotic scenario

$$n \to \infty, \qquad \Delta_n \to 0, \qquad n\Delta_n \to \infty$$

Gobet (2002): Under $P_{\alpha,\beta}$ the Fisher information is

$$\begin{pmatrix} i_1(\alpha,\beta) & 0 \\ 0 & i_2(\alpha,\beta) \end{pmatrix}$$

where

$$\begin{aligned} i_1(\alpha,\beta) &= \int_D \partial_\alpha b(x;\alpha)^\top V(x;\beta)^{-1} \partial_{\alpha^\top} b(x;\alpha) \nu_{\alpha,\beta}(x) dx \\ i_2(\alpha,\beta) &= \frac{1}{2} \int_D \partial_\beta \mathrm{vec}\,(V(x;\beta))^\top V^{\otimes 2}(x;\beta)^{-1} \partial_{\beta^\top} \mathrm{vec}\,(V(x;\beta)) \nu_{\alpha,\beta}(x) dx \end{aligned}$$

$\mathrm{vec}\,(V) = (V_{11}, V_{2,1}, \ldots, V_{d-1,d}, V_{dd})^\top \in \mathbb{R}^{d^2}$

$\otimes$ tensor/Kronecker product, $V^{\otimes 2}$ is a $d^2 \times d^2$-matrix



Conditions ensuring rate optimality and efficiency

High frequency/infinite horizon asymptotic scenario

$$n \to \infty, \qquad \Delta_n \to 0, \qquad n\Delta_n \to \infty$$

$$G_n^{LT}(\theta) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{j,\theta}(\Delta, X_{t_{i-1}}^{LT}) \left[\phi_{j,\theta}(X_{t_i}^{LT}) - e^{-\lambda_{j,\theta}\Delta} \phi_{j,\theta}(f_{\Delta,\theta}(X_{t_{i-1}}^{LT}))\right]$$

$$G_n^{S}(\theta) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{j,\theta}(\Delta, X_{t_{i-1}}^{S}) \left[\phi_{j,\theta}(f_{-\Delta/2,\theta}(X_{t_i}^{S})) - e^{-\lambda_{j,\theta}\Delta} \phi_{j,\theta}(f_{\Delta/2,\theta}(X_{t_{i-1}}^{S}))\right]$$

The following condition from Sørensen (2024) on the weights $a_{j,\theta}(\Delta, x) = (a_{1j,\theta}(\Delta, x), \ldots, a_{pj,\theta}(\Delta, x))^\top$ ensure rate optimality and efficiency for both G^{LT} and G^{S}.



Conditions ensuring rate optimality and efficiency

Condition 1

The weight functions $a_{j,\alpha,\beta} = (a_{1j,\alpha,\beta}, \ldots, a_{pj,\alpha,\beta})^\top$ satisfy the following for all $(\alpha, \beta) \in \Theta_1 \times \Theta_2$ and for all $x \in D$:

$$\sum_{j=1}^{m} a_{ij,\alpha,\beta}(0,x)\partial_x \phi_{j,\alpha,\beta}(x) = 0, \quad i = p_1 + 1, \ldots, p,$$

$$\sum_{j=1}^{m} a_{ij,\alpha,\beta}(0,x)\partial_x \phi_{j,\alpha,\beta}(x) = V(x;\beta)^{-1}\partial_{\alpha_i} F(x;\alpha), \quad i = 1, \ldots, p_1,$$

$$\sum_{j=1}^{m} a_{(p_1+i)j,\alpha,\beta}(0,x)\mathrm{vec}\left(\partial^2_{xx^\top}\phi_{j,\alpha,\beta}(x)\right) = V^{\otimes 2}(x;\beta)^{-1}\mathrm{vec}\left(\partial_{\beta_i} V(x;\beta)\right), \quad i = 1, \ldots, p_2$$

$\partial^2_{xx^\top}\phi_{j,\alpha,\beta}(x)$ is the Hessian of $\phi_{j,\alpha,\beta}(x)$

A solution exists if $m \geq d(d+3)/2$

For $d = 1$, a solution exists if $m \geq 2$



High frequency asymptotics: normality and efficiency

Theorems 3.2, 4.1 and 4.4 in Sørensen (2024) imply the following result

$\theta_0 = (\alpha_0, \beta_0)$ denotes the true parameter value

Theorem

Suppose Condition 1 holds (+ regularity conditions, including $i_j(\theta_0) > 0$, $j = 1, 2$, smoothness, integrability and polynomial growth). Assume, moreover, that the following identifyability condition holds

$$\int_D [F(x;\alpha_0) - F(x;\alpha)]^\top V(x;\beta)^{-1} \partial_{\alpha^\top} F(x;\alpha) \nu_{\theta_0}(x) dx \neq \mathbf{0}, \quad \text{when } \alpha \neq \alpha_0$$

$$\int_D \operatorname{vec}\left(V(x;\beta_0) - V(x;\beta)\right)^\top V^{\otimes 2}(x;\beta)^{-1} \partial_{\beta^\top} \operatorname{vec}\left(V(x;\beta)\right) \nu_{\theta_0}(x) dx \neq \mathbf{0}$$

when $\beta \neq \beta_0$

Then with a probability that goes to one as $n \to \infty$, a consistent G_n^{LT}-estimator $(\hat{\alpha}_n^{LT}, \hat{\beta}_n^{LT})$ and a consistent G_n^S-estimator $(\hat{\alpha}_n^S, \hat{\beta}_n^S)$ exist and both are unique in any compact subset K of Θ with $\theta_0 \in \mathit{int}\, K$.



High frequency asymptotics: normality and efficiency

Theorem (continued)

Moreover, if $n\Delta^4 \to 0$,

$$\begin{pmatrix} \sqrt{n\Delta_n}(\hat{\alpha}_n^S - \alpha_0) \\ \sqrt{n}(\hat{\beta}_n^S - \beta_0) \end{pmatrix} \xrightarrow{\mathcal{D}} N_p\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} i_1(\alpha_0,\beta_0)^{-1} & 0 \\ 0 & i_2(\alpha_0,\beta_0)^{-1} \end{pmatrix}\right)$$

The stronger assumption $n\Delta^2 \to 0$ is needed to obtain asymptotic normality for the G_n^{LT}-estimator:

$$\begin{pmatrix} \sqrt{n\Delta_n}(\hat{\alpha}_n^{LT} - \alpha_0) \\ \sqrt{n}(\hat{\beta}_n^{LT} - \beta_0) \end{pmatrix} \xrightarrow{\mathcal{D}} N_p\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} i_1(\alpha_0,\beta_0)^{-1} & 0 \\ 0 & i_2(\alpha_0,\beta_0)^{-1} \end{pmatrix}\right)$$

