Estimation for SPDEs in two space dimensions with unknown damping parameters using temporal and spatial increments

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Plan of today's talk

A linear parabolic stochastic partial differential equation (SPDE) in two space dimensions

Estimators for the damping parameters of the noise terms in the SPDE using high-frequency spatio-temporal data

Minimum contrast estimators and parametric adaptive estimators for unknown coefficient parameters of the SPDE

Asymptotic properties of the proposed estimators

An example and simulation results

Introduction

We consider the following linear parabolic stochastic partial differential equation (SPDE) in two space dimensions

$$\begin{aligned} \mathrm{d}X_t^Q(y,z) &= \left\{\theta_2\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\theta_1\frac{\partial}{\partial y}+\eta_1\frac{\partial}{\partial z}+\theta_0\right\}X_t^Q(y,z)\mathrm{d}t \\ &\quad +\sigma\mathrm{d}W_t^Q(y,z), \quad (t,y,z)\in[0,1]\times D \qquad (1) \\ X_0^Q(y,z) &= \xi(y,z), \quad (y,z)\in D, \\ X_t^Q(y,z) &= 0, \quad (t,y,z)\in[0,1]\times\partial D, \end{aligned}$$

where $D=(0,1)^2$,
W_t^Q is a Q-Wiener process with an unknown damping parameter α,
ξ is a deterministic initial value,
$\theta=(\theta_0,\theta_1,\eta_1,\theta_2)$ and σ are unknown parameters, $(\theta_0,\theta_1,\eta_1,\theta_2,\sigma)\in\Theta$,
the parameter space Θ is a compact convex subset of $\mathbb{R}^3\times(0,\infty)^2$.
Moreover, $\theta^*=(\theta_0^*,\theta_1^*,\eta_1^*,\theta_2^*)$ and σ^* are true values of the parameters and we assume that $(\theta^*,\sigma^*)\in\operatorname{Int}\Theta$.
The data are discrete observations $\mathbb{X}_{N,M}=\{X_{t_i}^Q(y_{j_1},z_{j_2})\}$, $i=0,\ldots,N$, $j_1=0,\ldots,M_1$, $j_2=0,\ldots,M_2$, $M=M_1M_2$ with $t_i=i\Delta$, $y_{j_1}=j_1/M_1$ and $z_{j_2}=j_2/M_2$, where $\Delta=1/N$.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a stochastic basis with usual conditions, and let $\{w_{l_1,l_2}\}_{l_1,l_2\geq 1}$ be independent real-valued standard Wiener processes on this basis.
Let A_θ be the differential operator defined by

$$-A_\theta = \theta_2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \theta_1\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial z} + \theta_0.$$

Notice that SPDE (1) can be expressed as

$$\mathrm{d}X_t^Q(y,z) = -A_\theta X_t^Q(y,z)\mathrm{d}t + \sigma\mathrm{d}W_t^Q(y,z).$$

For $l_1, l_2 \geq 1$ and $y, z \in [0,1]$, the eigenfunctions e_{l_1,l_2} of the differential operator A_θ and the corresponding eigenvalues λ_{l_1,l_2} are given by

$$e_{l_1}^{(1)}(y) = \sqrt{2}\sin(\pi l_1 y)\mathrm{e}^{-\frac{\theta_1}{2\theta_2}y}, \quad e_{l_2}^{(2)}(z) = \sqrt{2}\sin(\pi l_2 z)\mathrm{e}^{-\frac{\eta_1}{2\theta_2}z},$$

$$e_{l_1,l_2}(y,z) = e_{l_1}^{(1)}(y)e_{l_2}^{(2)}(z) = 2\sin(\pi l_1 y)\sin(\pi l_2 z)\mathrm{e}^{-\frac{\theta_1}{2\theta_2}y}\mathrm{e}^{-\frac{\eta_1}{2\theta_2}z},$$

$$\lambda_{l_1,l_2} = -\theta_0 + \frac{\theta_1^2+\eta_1^2}{4\theta_2} + \pi^2(l_1^2+l_2^2)\theta_2.$$

We then see that $A_\theta e_{l_1,l_2} = \lambda_{l_1,l_2}e_{l_1,l_2}$.
The eigenfunctions $\{e_{l_1,l_2}\}_{l_1,l_2\geq 1}$ are orthonormal with respect to the weighted L^2-inner product

$$\langle u, v\rangle := \langle u, v\rangle_\theta = \int_0^1\int_0^1 u(y,z)v(y,z)\mathrm{e}^{\frac{\theta_1}{\theta_2}y}\mathrm{e}^{\frac{\eta_1}{\theta_2}z}\mathrm{d}y\mathrm{d}z, \quad \|u\| = \sqrt{\langle u, u\rangle}.$$

Q_1-Wiener process

While two types of Q-Wiener processes, $W_t^{Q_1}$ and $W_t^{Q_2}$, can be considered, we will focus on the Q_1-Wiener process due to time constraints. The Q_1-Wiener process is given by

$$W_t^{Q_1}(y,z) = \sum_{l_1,l_2 \geq 1} \lambda_{l_1,l_2}^{-\alpha/2} e_{l_1,l_2}(y,z) w_{l_1,l_2}(t) \tag{2}$$

for $t \geq 0$, where $\{w_{l_1,l_2}\}_{l_1,l_2 \geq 1}$ are independent $\mathbb{R}$-valued standard Wiener processes, $\alpha \in (0,2)$ is unknown, $\alpha \in [\underline{\alpha}, \overline{\alpha}] \subset (0,2)$, the true value α^* belongs to $(\underline{\alpha}, \overline{\alpha})$ and $0 < \underline{\alpha} < \overline{\alpha} < 2$.

$$e_{l_1,l_2}(y,z) = 2\sin(\pi l_1 y)\sin(\pi l_2 z)\mathrm{e}^{-\frac{\theta_1}{2\theta_2}y}\mathrm{e}^{-\frac{\eta_1}{2\theta_2}z},$$

$$\lambda_{l_1,l_2} = -\theta_0 + \frac{\theta_1^2+\eta_1^2}{4\theta_2} + \pi^2(l_1^2+l_2^2)\theta_2$$

for $l_1, l_2 \geq 1$ and $y, z \in [0,1]$.

The condition $\alpha > 0$ ensures that $\sup_{0 \leq t \leq 1} \mathbb{E}[\|X_t^{Q_1}\|_\theta^2] < \infty$.

The condition $\alpha < 2$ guarantees that our proposed estimator given in (12) below is bounded in probability at the rate $\sqrt{mN} = O(N)$.

Refer to Bossert (2024) for the estimation of α.

Q_2-Wiener process

The Q_2-Wiener process $W_t^{Q_2}$ is given by

$$W_t^{Q_2}(y,z) = \sum_{l_1,l_2 \geq 1} \mu_{l_1,l_2}^{-\alpha/2} e_{l_1,l_2}(y,z) w_{l_1,l_2}(t) \tag{3}$$

for $t \geq 0$, where $\{w_{l_1,l_2}\}_{l_1,l_2 \geq 1}$ are independent $\mathbb{R}$-valued standard Wiener processes, $\alpha \in (0,2)$ is unknown, $\alpha \in [\underline{\alpha}, \overline{\alpha}] \subset (0,2)$, the true value α^* belongs to $(\underline{\alpha}, \overline{\alpha})$ and $0 < \underline{\alpha} < \overline{\alpha} < 2$.

$$e_{l_1,l_2}(y,z) = 2\sin(\pi l_1 y)\sin(\pi l_2 z)\mathrm{e}^{-\frac{\theta_1}{2\theta_2}y}\mathrm{e}^{-\frac{\eta_1}{2\theta_2}z},$$
$$\mu_{l_1,l_2} = \pi^2(l_1^2 + l_2^2) + \mu_0,$$

μ_0 is unknown, the parameter space of μ_0 is a compact convex subset of $(-2\pi^2, \infty)$ and the true value μ_0^* belongs to its interior.

In the case that the driving noise is the Q_2-Wiener process, θ_0 cannot be estimated, but $(\theta_1, \eta_1, \theta_2, \sigma^2, \mu_0, \alpha)$ can be estimated.

Due to time constraints, this talk will focus only on the SPDE model driven by the Q_1-Wiener process.

We make the following condition on the initial value ξ of SPDE (1).

[A1] The initial value $\xi \in L^2(D)$ is non-random and $\|A_\theta^{(1+\overline{\alpha})/2}\xi\| < \infty$.

Note that $\overline{\alpha}$ is the upper bound of the parameter space of α.

Moreover, we assume $\lambda_{1,1}^* > 0$.

For $u = \sum_{l_1,l_2\geq 1}\langle u, e_{l_1,l_2}\rangle e_{l_1,l_2}$, $A_\theta^\alpha u = \sum_{l_1,l_2\geq 1}\lambda_{l_1,l_2}^\alpha \langle u, e_{l_1,l_2}\rangle e_{l_1,l_2}$.

It is possible to consider the general initial condition as in Tonaki et al. (2023).

Q_1-Wiener process

For the Q_1-Wiener process $W_t^{Q_1}$, the random field $X_t^{Q_1}(y,z)$ is spectrally decomposed as

$$X_t^{Q_1}(y,z) = \sum_{l_1,l_2 \geq 1} x_{l_1,l_2}^{Q_1}(t) e_{l_1,l_2}(y,z), \quad t \geq 0,\ y,z \in [0,1],$$

where $e_{l_1,l_2}(y,z) = 2\sin(\pi l_1 y)\sin(\pi l_2 z)\mathrm{e}^{-\frac{\theta_1}{2\theta_2}y}\mathrm{e}^{-\frac{\eta_1}{2\theta_2}z}$, the coordinate process

$$x_{l_1,l_2}^{Q_1}(t) = \langle X_t^{Q_1}, e_{l_1,l_2}\rangle = \mathrm{e}^{-\lambda_{l_1,l_2}t}\langle \xi, e_{l_1,l_2}\rangle + \sigma\lambda_{l_1,l_2}^{-\alpha/2}\int_0^t \mathrm{e}^{-\lambda_{l_1,l_2}(t-s)}\mathrm{d}w_{l_1,l_2}(s), \quad (4)$$

$\lambda_{l_1,l_2} = -\theta_0 + \frac{\theta_1^2+\eta_1^2}{4\theta_2} + \pi^2(l_1^2+l_2^2)\theta_2$ and $\alpha \in (0,2)$ is an unknown parameter.

Note that $\{x_{l_1,l_2}^{Q_1}\}_{l_1,l_2\geq 1}$ are the Ornstein-Uhlenbeck processes defined by

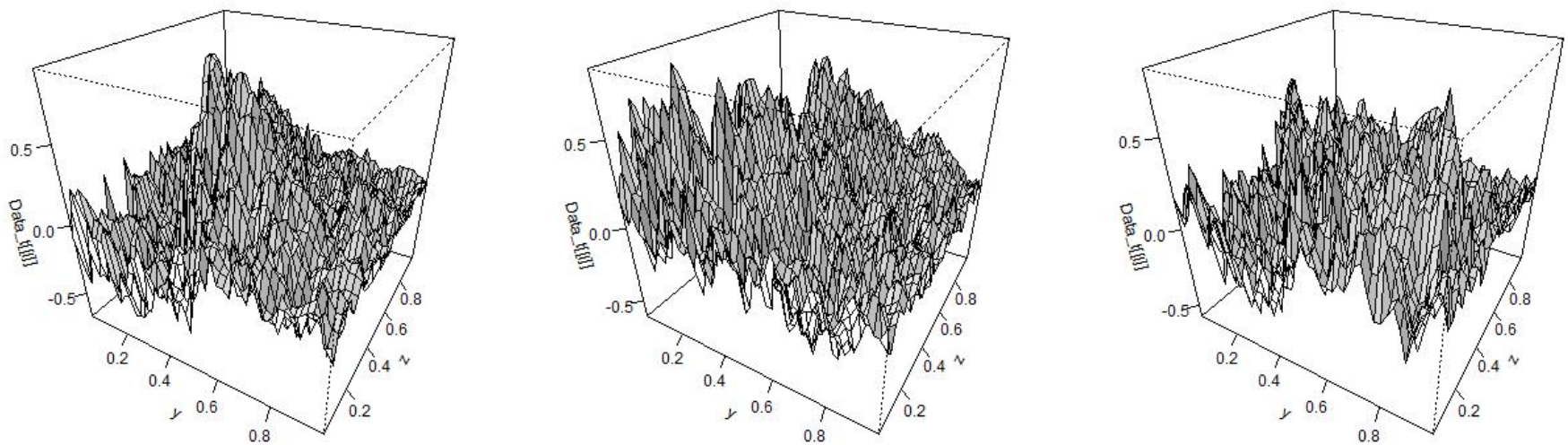
$$\mathrm{d}x_{l_1,l_2}^{Q_1}(t) = -\lambda_{l_1,l_2}x_{l_1,l_2}^{Q_1}(t)\mathrm{d}t + \sigma\lambda_{l_1,l_2}^{-\alpha/2}\mathrm{d}w_{l_1,l_2}(t), \quad x_{l_1,l_2}^{Q_1}(0) = \langle \xi, e_{l_1,l_2}\rangle \quad (5)$$

and are also expressed as

$$x_{l_1,l_2}^{Q_1}(t) = 2\int_0^1\int_0^1 X_t(y,z)\sin(\pi l_1 y)\sin(\pi l_2 z)\mathrm{e}^{\frac{\theta_1}{2\theta_2}y}\mathrm{e}^{\frac{\eta_1}{2\theta_2}z}\mathrm{d}y\mathrm{d}z.$$

Sample paths of SPDE: Q_1-model

In order to understand the characteristics of SPDE (1) driven by the Q_1-Wiener process (2), we will give some sample paths.
Figure 1 shows cross sections of sample paths of $X_t^{Q_1}(y,z)$ at $t = 0.1, 0.5$ and 0.9 when $(\theta_0^*, \theta_1^*, \eta_1^*, \theta_2^*, \sigma^*, \alpha^*) = (0, 0.2, 0.2, 0.2, 1, 0.5)$ and the initial condition $\xi = 0$.



$t = 0.1$ $t = 0.5$ $t = 0.9$

Figure 1: sample paths of $(\theta_0^*, \theta_1^*, \eta_1^*, \theta_2^*, \sigma^*, \alpha^*) = (0, 0.2, 0.2, 0.2, 1, 0.5)$

Remark: cylindrical Brownian motion

Consider the case where SPDE (1) is driven by a cylindrical Brownian motion $\{B_t\}_{t\geq 0}$ defined as

$$B_t(y,z) = \sum_{k,l\geq 1} e_{k,l}(y,z) w_{k,l}(t), \qquad t \geq 0, \tag{6}$$

that is, for the identity operator I and $B_t = W_t^I$, SPDE is represented by

$$\mathrm{d}X_t^I(y,z) = -A_\theta X_t^I(y,z)\mathrm{d}t + \sigma \mathrm{d}W_t^I(y,z). \tag{7}$$

In this case, the coordinate process $x_{k,l}^I(t)$ of SPDE (7) is expressed as

$$x_{k,l}^I(t) = \mathrm{e}^{-\lambda_{k,l}t}\langle \xi, e_{k,l}\rangle_\theta + \sigma \int_0^t \mathrm{e}^{-\lambda_{k,l}(t-s)} \mathrm{d}w_{k,l}(s).$$

Let $t_0 \in (0,1)$. Since there exists a constant $C_1 > 0$ such that $\lambda_{k,l} \leq C_1(k^2+l^2)$ for $k,l \geq 1$, it follows that for any $t \in [t_0, 1]$,

$$\begin{aligned}\mathbb{E}[\|X_t^I\|_\theta^2] = \sum_{k,l\geq 1} \mathbb{E}[x_{k,l}^I(t)^2] &= \sum_{k,l\geq 1} \mathrm{e}^{-2\lambda_{k,l}t}\mathbb{E}[\langle \xi, e_{k,l}\rangle_\theta^2] + \sigma^2 \sum_{k,l\geq 1} \frac{1-\mathrm{e}^{-2\lambda_{k,l}t}}{2\lambda_{k,l}} \\ &\geq \sigma^2 \sum_{k,l\geq 1} \frac{1-\mathrm{e}^{-2\lambda_{k,l}t}}{2\lambda_{k,l}} \geq C \sum_{k,l\geq 1} \frac{1}{\lambda_{k,l}} = \infty.\end{aligned}$$

This is a serious problem when we treat statistical inference for the SPDE, in particular, LS type estimation.

Q_1-Wiener process

For SPDE (1) driven by the Q_1-Wiener process defined as

$$W_t^{Q_1}(y,z) = \sum_{l_1,l_2 \geq 1} \lambda_{l_1,l_2}^{-\alpha/2} e_{l_1,l_2}(y,z) w_{l_1,l_2}(t) \tag{8}$$

for $t \geq 0$, where $\alpha \in (0,2)$ is an unknown constant,
$\lambda_{l_1,l_2} = -\theta_0 + \frac{\theta_1^2+\eta_1^2}{4\theta_2} + \pi^2(l_1^2+l_2^2)\theta_2$, $e_{l_1,l_2}(y,z) = 2\sin(\pi l_1 y)\sin(\pi l_2 z)\mathrm{e}^{-\frac{\theta_1}{2\theta_2}y}\mathrm{e}^{-\frac{\eta_1}{2\theta_2}z}$,
we have $\sup_{0\leq t\leq 1}\mathbb{E}[\|X_t^{Q_1}\|_\theta^2] < \infty$.
In fact, since there exists a constant $C_1 > 0$ such that $\lambda_{l_1,l_2} \geq C_1(l_1^2+l_2^2)$ for $l_1, l_2 \geq 1$, it holds that

$$\begin{aligned}
\mathbb{E}[\|X_t^{Q_1}\|_\theta^2] &= \sum_{l_1,l_2\geq 1} \mathbb{E}[x_{l_1,l_2}^{Q_1}(t)^2] \\
&= \sum_{l_1,l_2\geq 1} \mathrm{e}^{-2\lambda_{l_1,l_2}t}\langle \xi, e_{l_1,l_2}\rangle_\theta^2 + \sigma^2 \sum_{l_1,l_2\geq 1} \frac{1-\mathrm{e}^{-2\lambda_{l_1,l_2}t}}{2\lambda_{l_1,l_2}^{1+\alpha}} \\
&\leq \sum_{l_1,l_2\geq 1} \frac{1}{\lambda_{l_1,l_2}^{1+\overline{\alpha}}} \lambda_{l_1,l_2}^{1+\overline{\alpha}} \langle \xi, e_{l_1,l_2}\rangle_\theta^2 + \sigma^2 \sum_{l_1,l_2\geq 1} \frac{1}{2\lambda_{l_1,l_2}^{1+\alpha}} \\
&\leq C_1 + C_2 \sum_{l_1,l_2\geq 1} \frac{1}{\lambda_{l_1,l_2}^{1+\alpha}} < \infty
\end{aligned}$$

for $\alpha > 0$.

Previous study: Estimation of α (Bossert (2024))

Recently, Bossert (2024, SISP) proposed an estimator of α for the second-order linear parabolic SPDE in multiple space dimensions.

The estimator of $\alpha \in (0,1)$ for the second-order linear parabolic SPDE in two space dimensions is as follows:

$$\hat{\alpha}_{2n,m} = \frac{1}{\log(2)m} \sum_{j=1}^{m} \log\left(\frac{2RV_n(y_j, z_j)}{RV_{2n}(y_j, z_j)}\right),$$

where

$$RV_{2n}(y,z) = \sum_{i=1}^{2n} (X_{i\Delta_{2n}} - X_{(i-1)\Delta_{2n}})^2(y,z), \quad RV_n(y,z) = \sum_{i=1}^{n} (X_{i\Delta_n} - X_{(i-1)\Delta_n})^2(y,z).$$

He showed that $\sqrt{2nm_n}(\hat{\alpha}_{2n,m_{2n}} - \alpha)$ has asymptotic normality under $m_{2n} = O((2n)^\rho)$ for some $\rho \in (0, (1-\alpha)/4)$ and $2n = N$.

In this talk, we propose an estimator of α using triple increments, which involve both temporal and spatial increments.

Spatio-temporal data

We assume that a mild solution X is discretely observed on the grid $(t_i, y_j, z_k) \in [0,1]^3$ with

$$t_i = i\Delta, \quad y_j = j/M_1, \quad z_k = k/M_2$$

for $i = 0, \ldots, N$, $j = 0, \ldots, M_1$ and $k = 0, \ldots, M_2$, where $\Delta = 1/N$.
That is, the data are discrete observations
$\mathbb{X}_{\mathbf{M},N} = \{X_{t_i}(y_j, z_k); i = 0, \ldots, N, j = 0, \ldots, M_1, k = 0, \ldots, M_2\}$, where we set $\mathbf{M} = (M_1, M_2)$.

Thinned data

We consider SPDE (1) driven by the Q_1-Wiener process defined as

$$W_t^{Q_1}(y,z) = \sum_{l_1,l_2 \geq 1} \lambda_{l_1,l_2}^{-\alpha/2} e_{l_1,l_2}(y,z) w_{l_1,l_2}(t).$$

In order to obtain estimators of the damping parameter α and the coefficient parameters $(\theta_1, \eta_1, \theta_2, \sigma^2)$, we consider the thinned data of $\mathbb{X}_{\mathbf{M},N}$ as follows:

$$\mathbb{X}_{\mathbf{m},n}^{(b)} = \{X_{\widetilde{t}_i}(\widetilde{y}_j, \widetilde{z}_k); i = 0, \ldots, n, j = 0, \ldots, m_1, k = 0, \ldots, m_2\},$$

where $\mathbf{m} = (m_1, m_2)$, $m_1 = m_2$, $b \in [0, 1/2)$, $\delta = \frac{1-2b}{m_1}$,

$$\widetilde{t}_i = i\Delta_n := \left\lfloor \frac{N}{n} \right\rfloor \frac{i}{N}, \quad \widetilde{y}_j = b + j\delta, \quad \widetilde{z}_k = b + k\delta,$$

$\widetilde{y}_j \in \{y_0, \ldots, y_{M_1}\}$ and $\widetilde{z}_k \in \{z_0, \ldots, z_{M_2}\}$.

That is,

$$b \leq \widetilde{y}_0 < \widetilde{y}_1 < \cdots < \widetilde{y}_{m_1} \leq 1 - b, \quad b \leq \widetilde{z}_0 < \widetilde{z}_1 < \cdots < \widetilde{z}_{m_1} \leq 1 - b.$$

Note that it is necessary to choose (b, m_q) $(q = 1, 2)$ such that $(\widetilde{y}_j, \widetilde{z}_k)$ belongs to the original grids $\{y_0, \ldots, y_{M_1}\} \times \{z_0, \ldots, z_{M_2}\}$, for example, $bM_q \in \mathbb{N}$ and $\delta M_q = \frac{M_q}{m_q}(1 - 2b) \in \mathbb{N}$.

For a sequence $\{a_n\}$, we write $a_n \equiv a$ if $a_n = a$ for some $a \in \mathbb{R}$ and all n. For $h \in (0,1)$, $a, b, c > 0$ and $d \in \mathbb{R}$, we define

$$h^{a\widetilde{\wedge}b} = \begin{cases} h^a, & a < b, \\ -h^b \log(h), & a = b, \\ h^b, & a > b, \end{cases}$$

and $h^{d+c(a\widetilde{\wedge}b)} = h^d \cdot (h^c)^{a\widetilde{\wedge}b}$. Furthermore, for $L \in (1,\infty)$, we write

$$L^{a\widetilde{\wedge}b} = \frac{1}{(1/L)^{a\widetilde{\wedge}b}} = \begin{cases} L^a, & a < b, \\ L^b/\log(L), & a = b, \\ L^b, & a > b. \end{cases}$$

Estimation of α

For the temporal and spatial distances $\Delta = \frac{1}{N}$ and $\delta = \frac{1-2b}{m_1}$ between observations, we use two thinned data sets with the same ratio $\delta/\sqrt{\Delta}$.

Let $b \in (0, 1/2)$.

Suppose that we have two thinned data $\mathbb{X}^{(b)}_{\mathbf{m},N}$ and $\mathbb{X}^{(b)}_{\mathbf{m}',N'}$ obtained from $\mathbb{X}_{\mathbf{M},N}$, where $\mathbf{m}' = (m_1', m_2') = \mathbf{m}/2 = (m_1/2, m_2/2)$, $N' = N/4$, $m_1 = m_2$, $m = m_1 m_2 = \mathsf{O}(N)$ and $N = \mathsf{O}(m)$.

Note that $m_1' = \frac{m_1}{2}$,

$$\delta' := \frac{1-2b}{m_1'} = 2 \times \frac{1-2b}{m_1} = 2\delta, \quad \Delta' := \Delta_{N'} = \left\lfloor \frac{N}{N'} \right\rfloor \frac{1}{N} = 4 \times \frac{1}{N} = 4\Delta$$

and

$$\frac{\delta'}{\sqrt{\Delta'}} = \frac{\delta}{\sqrt{\Delta}}.$$

Triple increments

For the thinned data $\mathbb{X}^{(b)}_{\mathbf{m},N}$, we introduce the following temporal and spatial increments

$$\Delta_i X(y,z) = X_{t_i}(y,z) - X_{t_{i-1}}(y,z),$$
$$D_{j,k}X(t) = X_t(\widetilde{y}_j,\widetilde{z}_k) - X_t(\widetilde{y}_{j-1},\widetilde{z}_k) - \{X_t(\widetilde{y}_j,\widetilde{z}_{k-1}) - X_t(\widetilde{y}_{j-1},\widetilde{z}_{k-1})\}$$

and consider triple increments

$$\begin{aligned} T_{i,j,k}X &= (\Delta_i \circ D_{j,k})X = D_{j,k}X(t_i) - D_{j,k}X(t_{i-1}) \\ &= X_{t_i}(\widetilde{y}_j,\widetilde{z}_k) - X_{t_i}(\widetilde{y}_{j-1},\widetilde{z}_k) - \{X_{t_i}(\widetilde{y}_j,\widetilde{z}_{k-1}) - X_{t_i}(\widetilde{y}_{j-1},\widetilde{z}_{k-1})\} \\ &- \left(X_{t_{i-1}}(\widetilde{y}_j,\widetilde{z}_k) - X_{t_{i-1}}(\widetilde{y}_{j-1},\widetilde{z}_k) - \left\{X_{t_{i-1}}(\widetilde{y}_j,\widetilde{z}_{k-1}) - X_{t_{i-1}}(\widetilde{y}_{j-1},\widetilde{z}_{k-1})\right\}\right). \end{aligned}$$

Using the spectral decomposition $X_t^{Q_1}(y,z) = \sum_{l_1,l_2\geq 1} x^{Q_1}_{l_1,l_2}(t) e^{(1)}_{l_1}(y) e^{(2)}_{l_2}(z)$, we have

$$T_{i,j,k}X = \sum_{l_1,l_2\geq 1} (x_{l_1,l_2}(t_i) - x_{l_1,l_2}(t_{i-1}))(e^{(1)}_{l_1}(\widetilde{y}_j) - e^{(1)}_{l_1}(\widetilde{y}_{j-1}))(e^{(2)}_{l_2}(\widetilde{z}_k) - e^{(2)}_{l_2}(\widetilde{z}_{k-1})), \tag{9}$$

where $e^{(1)}_{l_1}(y) = \sqrt{2}\sin(\pi l_1 y)\mathrm{e}^{-\frac{\theta_1}{2\theta_2}y}$, $e^{(2)}_{l_2}(z) = \sqrt{2}\sin(\pi l_2 z)\mathrm{e}^{-\frac{\eta_1}{2\theta_2}z}$.

Q_1-Wiener process

We make the following condition of the initial value X_0 of SPDE (1).

[A1] The initial value $X_0 \in L^2(D)$ is deterministic and $\|A_\theta^{(1+\overline{\alpha})/2} X_0\| < \infty$.

Note that $\overline{\alpha}$ is the upper bound of the parameter space of α.

Let J_0 be the Bessel function of the first kind of order 0:

$$J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

For $r, \alpha > 0$, we set

$$\psi_{r,\alpha}(\theta_2) = \frac{2}{\theta_2 \pi} \int_0^\infty \frac{1 - \mathrm{e}^{-x^2}}{x^{1+2\alpha}} \left(J_0\left(\frac{\sqrt{2} r x}{\sqrt{\theta_2}}\right) - 2 J_0\left(\frac{r x}{\sqrt{\theta_2}}\right) + 1 \right) \mathrm{d}x. \tag{10}$$

Let $\kappa = \frac{\theta_1}{\theta_2}$, $\eta = \frac{\eta_1}{\theta_2}$. For $r, \alpha > 0$ and $\vartheta = (\kappa, \eta, \theta_2, \sigma^2)$, we define

$$g_{r,\alpha}(\vartheta) = \frac{\sigma^2 \psi_{r,\alpha}(\theta_2)}{(1-2b)^2} \int_b^{1-b} \int_b^{1-b} \mathrm{e}^{-\kappa y - \eta z} \mathrm{d}y \mathrm{d}z. \tag{11}$$

Q_1-Wiener process

For the triple increments $T_{i,j,k}X$ obtained from $\mathbb{X}^{(b)}_{\mathbf{m},N}$, we obtain for $r \equiv \delta/\sqrt{\Delta} \in (0,\infty)$,

$$\frac{1}{mN\Delta^{\alpha}} \sum_{k=1}^{m_2} \sum_{j=1}^{m_1} \sum_{i=1}^{N} (T_{i,j,k}X)^2 \xrightarrow{p} g_{r,\alpha}(\vartheta)$$

under [A1].

Let $T'_{i,j,k}X$ be the triple increments obtained from $\mathbb{X}^{(b)}_{\mathbf{m}',N'}$. Similarly, we have for $r' \equiv \delta'/\sqrt{\Delta'} \in (0,\infty)$,

$$\frac{1}{m'N'(\Delta')^{\alpha}} \sum_{k=1}^{m'_2} \sum_{j=1}^{m'_1} \sum_{i=1}^{N'} (T'_{i,j,k}X)^2 \xrightarrow{p} g_{r',\alpha}(\vartheta)$$

under [A1].

Since $r' = \delta'/\sqrt{\Delta'} = \delta/\sqrt{\Delta} = r$, we obtain

$$\frac{\dfrac{1}{m'N'(\Delta')^{\alpha}} \displaystyle\sum_{k=1}^{m'_2} \sum_{j=1}^{m'_1} \sum_{i=1}^{N'} (T'_{i,j,k}X)^2}{\dfrac{1}{mN\Delta^{\alpha}} \displaystyle\sum_{k=1}^{m_2} \sum_{j=1}^{m_1} \sum_{i=1}^{N} (T_{i,j,k}X)^2} \xrightarrow{p} 1.$$

Since it follows that

$$\frac{\frac{1}{m'N'(\Delta')^{\alpha}}\sum_{k=1}^{m_2'}\sum_{j=1}^{m_1'}\sum_{i=1}^{N'}(T'_{i,j,k}X)^2}{\frac{1}{mN\Delta^{\alpha}}\sum_{k=1}^{m_2}\sum_{j=1}^{m_1}\sum_{i=1}^{N}(T_{i,j,k}X)^2} = \frac{\Delta^{\alpha}}{(\Delta')^{\alpha}} \times \frac{\frac{1}{m'N'}\sum_{k=1}^{m_2'}\sum_{j=1}^{m_1'}\sum_{i=1}^{N'}(T'_{i,j,k}X)^2}{\frac{1}{mN}\sum_{k=1}^{m_2}\sum_{j=1}^{m_1}\sum_{i=1}^{N}(T_{i,j,k}X)^2}$$

$$= \frac{1}{4^{\alpha}} \times \frac{\frac{1}{m'N'}\sum_{k=1}^{m_2'}\sum_{j=1}^{m_1'}\sum_{i=1}^{N'}(T'_{i,j,k}X)^2}{\frac{1}{mN}\sum_{k=1}^{m_2}\sum_{j=1}^{m_1}\sum_{i=1}^{N}(T_{i,j,k}X)^2},$$

we have

$$\frac{\frac{1}{m'N'}\sum_{k=1}^{m_2'}\sum_{j=1}^{m_1'}\sum_{i=1}^{N'}(T'_{i,j,k}X)^2}{\frac{1}{mN}\sum_{k=1}^{m_2}\sum_{j=1}^{m_1}\sum_{i=1}^{N}(T_{i,j,k}X)^2} \xrightarrow{p} 4^{\alpha}.$$

We thus define the estimator of the damping parameter α as follows.

$$\widehat{\alpha} = \log \left(\frac{\frac{1}{m'N'} \sum_{k=1}^{m_2'} \sum_{j=1}^{m_1'} \sum_{i=1}^{N'} (T'_{i,j,k} X)^2}{\frac{1}{mN} \sum_{k=1}^{m_2} \sum_{j=1}^{m_1} \sum_{i=1}^{N} (T_{i,j,k} X)^2} \right) / \log(4). \tag{12}$$

We then obtain the following result.

Theorem 1

Under [A1], as $m \to \infty$ and $N \to \infty$,

$$\sqrt{mN}(\widehat{\alpha} - \alpha^*) = \mathsf{O}_p(1).$$

Remark 1

(i) *This result also holds true for the Q_2-model without changing the form of the estimator $\widehat{\alpha}$.*

(ii) *In the above, we chose the thinned data such that $\delta' = 2\delta$ and $\Delta' = 4\Delta$, but an estimator of α can be obtained even if we choose $\delta' = p\delta$ and $\Delta' = p^2\Delta$ for $p = 3, 4, \ldots$. In this case, the estimator of α is given by replacing $\log(4)$ in (12) with $\log(p^2)$.*

(iii) *Bossert (2024) proposed the estimator of α of the parabolic SPDEs in d-space dimensions based on temporal realized quadratic variations, and in particular for $d = 2$, showed that the estimator has asymptotic normality with the convergence rate $\sqrt{mN} = \mathrm{O}(N^{(1+\rho)/2})$ for some $\rho \in (0, \frac{1-\alpha}{4})$ ($\alpha \in (0, 1)$). Since we use temporal and spatial realized quadratic variations to estimate α, we can estimate α over the wider range $(0, 2)$ with the faster convergence rate $\sqrt{mN} = \mathrm{O}(N)$ than Bossert (2024).*

(iv) *Since the estimator $\widehat{\alpha}$ given in (12) is constructed from the spatio-temporal realized quadratic variation $\displaystyle\sum_{k=1}^{m_2}\sum_{j=1}^{m_1}\sum_{i=1}^{N}(T_{i,j,k}X)^2$, the asymptotic properties of*

$$S_{j,k} = \frac{1}{N\Delta^{\alpha}}\sum_{i=1}^{N}(T_{i,j,k}X)^2$$

are inherited by $\widehat{\alpha}$. Tonaki et al. (2025a) studied the asymptotic properties of $S_{j,k}$ and showed that for the Q_1-model,

$$\mathbb{E}[S_{j,k}] = \sigma^2 \mathrm{e}^{-\kappa(\widetilde{y}_{j-1}+\widetilde{y}_j)/2-\eta(\widetilde{z}_{k-1}+\widetilde{z}_k)/2}\psi_{r,\alpha}(\theta_2) + \mathrm{O}(\Delta),$$

where $\psi_{r,\alpha}$ is given by (10) (see Proposition 2.1 in Tonaki et al. (2025a)). As it is noted in Remark 1-(ii) of Tonaki et al. (2025a), it is challenging to show the asymptotic normality of the estimator $\widehat{\alpha}$ because one needs a stricter error estimate to obtain more precise asymptotic properties of $S_{j,k}$. This is future work.

Estimation for the coefficient parameters of Q_1-model

We consider minimum contrast estimators of the coefficient parameters in SPDE (1) with an unknown damping parameter α using the methodology proposed by Tonaki et al. (2025a). Let $\kappa = \frac{\theta_1}{\theta_2}$, $\eta = \frac{\eta_1}{\theta_2}$ and $b \in (0, 1/2)$.

Suppose that we have the thinned data $\mathbb{X}^{(b)}_{\mathbf{m},N}$ obtained from $\mathbb{X}_{\mathbf{M},N}$, where $m = m_1 m_2$, $N = \mathsf{O}(m)$ and $m = \mathsf{O}(N)$. We write

$$\overline{y}_j = \frac{\widetilde{y}_{j-1} + \widetilde{y}_j}{2}, \quad \overline{z}_k = \frac{\widetilde{z}_{k-1} + \widetilde{z}_k}{2}, \quad j = 1, \ldots, m_1, k = 1, \ldots, m_2.$$

Let J_0 be the Bessel function of the first kind of order 0: $J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$.

For $r, \alpha > 0$, we set $\psi_{r,\alpha}(\theta_2) = \frac{2}{\theta_2 \pi} \int_0^\infty \frac{1 - \mathrm{e}^{-x^2}}{x^{1+2\alpha}} \left(J_0\left(\frac{\sqrt{2} r x}{\sqrt{\theta_2}}\right) - 2 J_0\left(\frac{r x}{\sqrt{\theta_2}}\right) + 1 \right) \mathrm{d}x$.

Proposition 1

Let $\alpha \in (0, 2)$ and $\delta/\sqrt{\Delta} \equiv r \in (0, \infty)$. Under [A1], it holds that

$$\mathbb{E}[(T_{i,j,k} X)^2] = \Delta^\alpha \sigma^2 \mathrm{e}^{-\kappa \overline{y}_j} \mathrm{e}^{-\eta \overline{z}_k} \psi_{r,\alpha}(\theta_2) + R_{i,j,k} + O(\Delta^{1+\alpha}),$$

where $\sum_{i=1}^N R_{i,j,k} = O(\Delta^\alpha)$ uniformly in j, k.

Let $\widetilde{T}_{i,j,k}X = T_{i,j,k}X + T_{i+1,j,k}X$ and $\vartheta = (\kappa, \eta, \theta_2, \sigma^2)$. We define

$$\mathcal{K}_{\mathbf{m},N}(\vartheta;\alpha) = \frac{1}{m}\sum_{k=1}^{m_2}\sum_{j=1}^{m_1}\left\{\frac{1}{N\Delta^\alpha}\sum_{i=1}^{N}(T_{i,j,k}X)^2 - f_{r,\alpha}(\overline{y}_j,\overline{z}_k;\vartheta)\right\}^2$$
$$+\frac{1}{m}\sum_{k=1}^{m_2}\sum_{j=1}^{m_1}\left\{\frac{1}{N(2\Delta)^\alpha}\sum_{i=1}^{N-1}(\widetilde{T}_{i,j,k}X)^2 - f_{r/\sqrt{2},\alpha}(\overline{y}_j,\overline{z}_k;\vartheta)\right\}^2,$$

where $f_{r,\alpha}(y,z;\vartheta) = \sigma^2 \mathrm{e}^{-\kappa y-\eta z}\psi_{r,\alpha}(\theta_2)$.
Let Ξ be the parameter space of ϑ and a compact convex subset of $\mathbb{R}^2 \times (-\frac{r^2}{8\log(\sqrt{2}-1)}, \infty) \times (0,\infty)$. We assume that the true value $\vartheta^* = (\kappa^*, \eta^*, \theta_2^*, (\sigma^*)^2)$ belongs to $\mathrm{Int}(\Xi)$. For the estimator $\widehat{\alpha}$ in (12), we define the estimator of $\vartheta = (\kappa, \eta, \theta_2, \sigma^2)$ by

$$\widehat{\vartheta} = \operatorname*{argmin}_{\vartheta\in\Xi} \mathcal{K}_{\mathbf{m},N}(\vartheta;\widehat{\alpha}).$$

We then obtain the following result.

Theorem 2

Under [A1], as $m \to \infty$ and $N \to \infty$,

$$\frac{\sqrt{mN}}{\log(N)}(\widehat{\vartheta} - \vartheta^*) = \mathsf{O}_p(1).$$

Remark 2

(i) *In order for* θ_2 *to be identifiable, the function* $\theta_2 \mapsto G_{r,\alpha}(\theta_2) = \psi_{r,\alpha}(\theta_2)/\psi_{r/\sqrt{2},\alpha}(\theta_2)$ *needs to be injective for* $\theta_2 \in (0, \infty)$ *(see Subsection 4.6 in Tonaki et al. (2025a)).*

Tonaki et al. (2025a) showed that $G_{r,\alpha}(\theta_2)$ *is injective for* $\theta_2 \in (0, \infty)$ *when* $\alpha \in [1, 2)$ *(see Lemma 4.12 in Tonaki et al. (2025a)), but when* $\alpha \in (0, 1)$*, they showed only that* $G_{r,\alpha}(\theta_2)$ *is injective for* $\theta_2 \in (-\frac{r^2}{8\log(2^{\alpha/2}-1)}, \infty)$ *due to technical difficulties (see Lemma 4.13 in Tonaki et al. (2025a)).*

For that reason, to guarantee identifiability of θ_2*, they provided the following lower bound of its parameter space.*

$$\underline{\theta_2}(r,\alpha) = \begin{cases} -\frac{r^2}{8\log(2^{\alpha/2}-1)}, & \alpha \in (0,1), \\ 0, & \alpha \in [1,2). \end{cases}$$

Since the lower bound $\underline{\theta_2}(r, \alpha)$ *relies on* α*, and* $\alpha \in (0, 2)$ *is an unknown parameter in this study, the lower bound of the parameter space for* θ_2 *is defined as*

$$\sup_{\alpha\in(0,2)} \underline{\theta_2}(r,\alpha) = -\frac{r^2}{8\log(\sqrt{2}-1)}$$

to ensure its identifiability. That is, we set the parameter space Ξ *so that the parameter* ϑ *is identifiable for any* $\alpha \in (0, 2)$*.*

(ii) *Since* $N = \mathsf{O}(m)$ *and* $m = \mathsf{O}(N)$*, we have*

$$\frac{\sqrt{mN}}{\log(\sqrt{mN})}(\widehat{\vartheta} - \vartheta^*) = \mathsf{O}_p(1).$$

Remark 3

(iii) *For the estimator* $\widehat{\vartheta} = (\widehat{\kappa}, \widehat{\eta}, \widehat{\theta}_2, \widehat{\sigma}^2)$, *we can get the estimators of the coefficient parameters* θ_1 *and* η_1 *by* $\widehat{\theta}_1 = \widehat{\kappa}\widehat{\theta}_2$ *and* $\widehat{\eta}_1 = \widehat{\eta}\widehat{\theta}_2$, *respectively. Therefore, the estimator* $(\widehat{\theta}_1, \widehat{\eta}_1, \widehat{\theta}_2, \widehat{\sigma}^2)$ *is bounded in probability at the rate* $\sqrt{mN}/\log(N)$. *Note that when* α *is known,* $(\widehat{\theta}_1, \widehat{\eta}_1, \widehat{\theta}_2, \widehat{\sigma}^2)$ *is bounded in probability at the rate* $\sqrt{mN}$.

(iv) *In order to obtain the convergence rates of the estimators* $(\widehat{\theta}_1, \widehat{\eta}_1, \widehat{\theta}_2, \widehat{\sigma}^2)$, *we need to control the difference between* $\mathcal{K}_{\mathbf{m},N}(\vartheta; \widehat{\alpha})$ *and* $\mathcal{K}_{\mathbf{m},N}(\vartheta; \alpha^*)$, *which means that*

$$\left| \frac{1}{N\Delta^{\widehat{\alpha}}} \sum_{i=1}^{N} (T_{i,j,k}X)^2 - \frac{1}{N\Delta^{\alpha^*}} \sum_{i=1}^{N} (T_{i,j,k}X)^2 \right|$$

$$= \Delta^{\alpha^*} \left| \frac{1}{\Delta^{\widehat{\alpha}}} - \frac{1}{\Delta^{\alpha^*}} \right| \frac{1}{N\Delta^{\alpha^*}} \sum_{i=1}^{N} (T_{i,j,k}X)^2,$$

$$\Delta^{\alpha^*} \left| \frac{1}{\Delta^{\widehat{\alpha}}} - \frac{1}{\Delta^{\alpha^*}} \right| = \mathsf{O}_p\left(\frac{1}{\sqrt{mN}} \log\left(\frac{1}{\Delta} \right) \right) = \mathsf{O}_p\left(\frac{\log(N)}{\sqrt{mN}} \right).$$

Theorem 2 shows that the convergence rates of the estimators $(\widehat{\theta}_1, \widehat{\eta}_1, \widehat{\theta}_2, \widehat{\sigma}^2)$ *are reduced by* $1/\log(N)$ *when* α *is unknown, compared to when* α *is known.*

We next construct the estimator of θ_0 with asymptotic normality at the rate $\sqrt{n}$ using the following temporal thinned data $\mathbb{X}^{(0)}_{\mathbf{M},n}$

$$\mathbb{X}^{(0)}_{\mathbf{M},n} = \{X_{\widetilde{t}_i}(y_j, z_k); i = 0, \ldots, n, j = 0, \ldots, M_1, k = 0, \ldots, M_2\},$$

where $n \leq N$ and $\widetilde{t}_i = \lfloor \frac{N}{n} \rfloor \frac{i}{N}$ for $i = 0, \ldots, n$.

Note that $\kappa = \frac{\theta_1}{\theta_2}$, $\eta = \frac{\eta_1}{\theta_2}$ and

$$x_{l_1,l_2}(t) = 2 \int_0^1 \int_0^1 X_t(y, z) \sin(\pi l_1 y) \sin(\pi l_2 z) \mathrm{e}^{\kappa y/2} \mathrm{e}^{\eta z/2} \mathrm{d}y \mathrm{d}z.$$

Using the operator $\Psi_{\mathbf{M}}$ defined by $\Psi_{\mathbf{M}} f(y, z) = f(y_{j-1}, z_{k-1})$ for $(y, z) \in [y_{j-1}, y_j) \times [z_{k-1}, z_k)$, $j = 1, \ldots, M_1$, $k = 1, \ldots, M_2$ and $f \in C(D)$, we have an approximate coordinate process $\widehat{x}_{l_1,l_2}(t)$ as follows.

$$\begin{aligned}\widehat{x}_{l_1,l_2}(t) &= 2 \int_0^1 \int_0^1 \Psi_{\mathbf{M}} X_t(y, z) \sin(\pi l_1 y) \sin(\pi l_2 z) \mathrm{e}^{(\widehat{\kappa} y + \widehat{\eta} z)/2} \mathrm{d}y \mathrm{d}z \\ &= \sum_{j=1}^{M_1} \sum_{k=1}^{M_2} X_t(y_{j-1}, z_{k-1}) \delta_j^{[y]} g_{l_1}(\widehat{\kappa}) \delta_k^{[z]} g_{l_2}(\widehat{\eta}),\end{aligned}$$

where

$$g_l(x : a) = \frac{\sqrt{2} \mathrm{e}^{ax/2}}{(a/2)^2 + (\pi l)^2} \left(\frac{a}{2} \sin(\pi l x) - \pi l \cos(\pi l x) \right), \quad a, x \in \mathbb{R},\ l \in \mathbb{N},$$

$$\delta_j^{[y]} g_l(a) = g_l(y_j : a) - g_l(y_{j-1} : a), \quad \delta_k^{[z]} g_l(a) = g_l(z_k : a) - g_l(z_{k-1} : a).$$

Note that the coordinate processes $\{x_{l_1,l_2}\}_{l_1,l_2\geq 1}$ are one-dimensional independent processes satisfying the Ornstein-Uhlenbeck dynamics:

$$\mathrm{d}x_{l_1,l_2}(t) = -\lambda_{l_1,l_2}x_{l_1,l_2}(t)\mathrm{d}t + \sigma\lambda_{l_1,l_2}^{-\alpha/2}\mathrm{d}w_{l_1,l_2}(t), \quad x_{l_1,l_2}(0) = \langle \xi, e_{l_1,l_2}\rangle. \tag{13}$$

Set $\sigma_{l_1,l_2} := \sigma\lambda_{l_1,l_2}^{-\alpha/2}$ in (13).

Note that $\lambda_{l_1,l_2} = -\theta_0 + \frac{\theta_1^2+\eta_1^2}{4\theta_2} + \pi^2(l_1^2+l_2^2)\theta_2 = -\theta_0 + \left(\frac{\kappa^2+\eta^2}{4} + \pi^2(l_1^2+l_2^2)\right)\theta_2$.

We define the estimator of σ_{l_1,l_2}^2 as

$$\tilde{\sigma}_{l_1,l_2}^2 = \sum_{i=1}^{n}(\hat{x}_{l_1,l_2}(\widetilde{t}_i) - \hat{x}_{l_1,l_2}(\widetilde{t}_{i-1}))^2.$$

We propose the adaptive estimators of λ_{l_1,l_2} and θ_0 as follows.

$$\tilde{\lambda}_{l_1,l_2} = \left(\frac{\hat{\sigma}^2}{\tilde{\sigma}_{l_1,l_2}^2}\right)^{1/\hat{\alpha}}, \quad \hat{\theta}_0 = -\tilde{\lambda}_{1,1} + \left(\frac{\hat{\kappa}^2+\hat{\eta}^2}{4} + 2\pi^2\right)\hat{\theta}_2.$$

We consider the following balance condition on n and $M_1 \wedge M_2$.

[A2]$_\alpha$ $\frac{n}{(M_1 \wedge M_2)^{2(\alpha \widetilde{\wedge} 1)}} \to 0$ as $n \to \infty$ and $M_1 \wedge M_2 \to \infty$.

[A3]$_\alpha$ $\frac{n}{(M_1 \wedge M_2)^{\alpha \widetilde{\wedge} 1}} \to 0$ as $n \to \infty$ and $M_1 \wedge M_2 \to \infty$.

Theorem 3

Under [A1], the followings hold.

(i) if [A2]$_{\alpha^*}$ *holds, then* $\hat{\theta}_0 \xrightarrow{p} \theta_0^*$ *as* $M_1 \wedge M_2 \to \infty$ *and* $n \to \infty$.

(ii) if [A3]$_{\alpha^*}$ *holds, then as* $M_1 \wedge M_2 \to \infty$ *and* $n \to \infty$,

$$\sqrt{n}(\hat{\theta}_0 - \theta_0^*) \xrightarrow{d} N(0, 2(\alpha^*)^{-2}(\lambda_{1,1}^*)^2).$$

Remark 4

(i) Since the true value α^* is unknown, we have no way to directly check the sufficient conditions [A2]$_{\alpha^*}$ and [A3]$_{\alpha^*}$ in Theorem 3.

In practice, it is natural to assume that [A2]$_\alpha$ (or [A3]$_\alpha$) holds for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, in which case we can check [A2]$_{\underline{\alpha}}$ (or [A3]$_{\underline{\alpha}}$).

(ii) Even when α is unknown, the estimators of the coefficient parameters based on the approximate coordinate process attain the same convergence rates as in the case when α is known (Theorem 2.3 in Tonaki et al. (2025a)).

Example and simulations: Q_1-model

The numerical solution of SPDE (1) with the Q_1-Wiener process is generated by

$$\bar{X}_{t_i}(y_j, z_k) = \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \bar{x}^{Q_1}_{l_1,l_2}(t_i) e_{l_1,l_2}(y_j, z_k), \tag{14}$$

for $i = 1, \ldots, N, j = 1, \ldots, M_1, k = 1, \ldots, M_2$, where

$$\bar{x}^{Q_1}_{l_1,l_2}(t_i) = \mathrm{e}^{-\lambda_{l_1,l_2}/N} \bar{x}^{Q_1}_{l_1,l_2}(t_{i-1}) + \sigma \lambda^{-\alpha/2}_{l_1,l_2} \int_{t_{i-1}}^{t_i} \mathrm{e}^{-\lambda_{l_1,l_2}(t_i - s)} \mathrm{d}w_{l_1,l_2}(s),$$

$\lambda_{l_1,l_2} = -\theta_0 + \frac{\theta_1^2 + \eta_1^2}{4\theta_2} + \pi^2(l_1^2 + l_2^2)\theta_2$ and

$e_{l_1,l_2}(y, z) = 2 \sin(\pi l_1 y) \sin(\pi l_2 z) \mathrm{e}^{-\frac{\theta_1}{2\theta_2} y} \mathrm{e}^{-\frac{\eta_1}{2\theta_2} z}$

In this simulation, the true values of parameters

$(\theta_0^*, \theta_1^*, \eta_1^*, \theta_2^*, \sigma^*, \alpha^*) = (0, 0.2, 0.2, 0.2, 1, 0.5).$

We set $N = 10^3$, $M_1 = M_2 = 200$, $L_1 = L_2 = 10^4$, $X_0 = 0$, $\lambda^*_{1,1} \approx 4.05$.

The number of iterations is 200.

The computation time of (14) is directly proportional to $L_1 \times L_2$.

Therefore, the computation time for the numerical solution of SPDE (1) is directly proportional to $N \times M_1 \times M_2 \times L_1 \times L_2$.

In the setting of this simulation, $N \times M_1 \times M_2 \times L_1 \times L_2 = 4 \times 10^{15}$.

When $N = 10^3$, $M_1 = M_2 = 200$, $L_1 = L_2 = 10^4$, the size of data is about 10 GB.

Three personal computers with Intel Gold 6128 (3.4 GHz) were used for this simulation, and it takes about 100 hours to generate one sample path of SPDE (1).

Table 1: Simulation results of $\widehat{\alpha}$, $\widehat{\theta}_0$, $\widehat{\theta}_1$, $\widehat{\eta}_1$, $\widehat{\theta}_2$ and $\widehat{\sigma}^2$.

		$\widehat{\alpha}$	$\widehat{\theta}_0$	$\widehat{\theta}_1$	$\widehat{\eta}_1$	$\widehat{\theta}_2$	$\widehat{\sigma}^2$
	true value	0.5	0	0.2	0.2	0.2	1
Case 1	mean	0.496	-0.362	0.202	0.202	0.205	0.997
	s.d.	(0.001)	(0.959)	(0.002)	(0.002)	(0.002)	(0.007)
Case 2	mean	0.482	-0.283	0.221	0.221	0.223	0.991
	s.d.	(0.002)	(1.090)	(0.005)	(0.005)	(0.004)	(0.020)
Case 3	mean	0.465	-1.046	0.240	0.240	0.242	0.977
	s.d.	(0.005)	(1.140)	(0.011)	(0.011)	(0.011)	(0.045)

Table 1 shows the simulation results of the means and the standard deviations (s.d.s) of $\hat{\alpha}$, $\widehat{\theta}_0$, $\widehat{\theta}_1$, $\widehat{\eta}_1$, $\widehat{\theta}_2$ and $\widehat{\sigma}^2$.

For the estimation of α, we set $(m_1, m_2, b) = (200, 200, 0.005)$, $(100, 100, 0.01)$ and $(50, 50, 0.02)$ in Cases 1, 2 and 3, respectively.

For the estimation of θ_0, θ_1, η_1, θ_2 and σ^2, we set $(n, m_1, m_2) = (100, 30, 30)$ in each case.

It seems from Table 1 that the biases of $\widehat{\alpha}$ are small and decrease as m_1 and m_2 increase. Moreover, Table 1 indicates that in Case 1, the biases of $\widehat{\theta}_1$, $\widehat{\eta}_1$, $\widehat{\theta}_2$ and $\widehat{\sigma}^2$ are very small and $\widehat{\theta}_0$ has a small bias.

Concluding remarks

$$\mathrm{d}X_t^Q(y,z) = \left\{\theta_2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \theta_1\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial z} + \theta_0\right\}X_t^Q(y,z)\mathrm{d}t$$
$$+ \sigma\mathrm{d}W_t^Q(y,z), \quad (t,y,z) \in [0,1]\times D,$$
$$X_0^Q(y,z) = \xi(y,z), \quad (y,z)\in D, \quad X_t^Q(y,z) = 0, \quad (t,y,z)\in[0,1]\times\partial D, \quad D = (0,1)^2.$$

In this talk, we have proposed
(i) an estimator for the damping parameter α,
(ii) a minimum contrast estimator (MCE) for (θ_2, θ_1, η_1, σ^2),
(iii) a parametric adaptive estimator (AE) for θ_0
of the linear parabolic SPDE in two space dimensions driven by the Q_1-Wiener process based on high-frequency spatio-temporal data.

The estimator $\widehat{\alpha}$ has N-consistency under some regularity conditions, where N is the number of time points in the spatio-temporal data.

The MCE $(\widehat{\theta}_1, \widehat{\eta}_1, \widehat{\theta}_2, \widehat{\sigma}^2)$ has $\frac{N}{\log(N)}$-consistency under some regularity conditions.

The AE $\widehat{\theta}_0$ has asymptotic normality under some regularity conditions, where $\widehat{\theta}_0 = -\widetilde{\lambda}_{1,1} + \frac{\widehat{\theta}_1^2+\widehat{\eta}_1^2}{4\widehat{\theta}_2} + 2\pi^2\widehat{\theta}_2$, $\widetilde{\lambda}_{1,1} = \left(\frac{\widehat{\sigma}^2}{\widetilde{\sigma}_{1,1}^2}\right)^{1/\widehat{\alpha}}$, $\widetilde{\sigma}_{1,1}^2 = \sum_{i=1}^n(\widehat{x}_{1,1}(\widetilde{t}_i) - \widehat{x}_{1,1}(\widetilde{t}_{i-1}))^2$.

We see from the simulation results that the proposed estimators work well.