Learning from MOM's principles

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(joint works with Geoffrey Chinot, Matthieu Lerasle and Timothée Mathieu)

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Test of robustness of the LASSO

$$Y = \langle X, t^* \rangle + \mathcal{N}(0,1) \text{ and } (X_1, Y_1), \cdots, (X_N, Y_N) \overset{i.i.d.}{\sim} (X, Y)$$

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$\cdots$	$\cdots$
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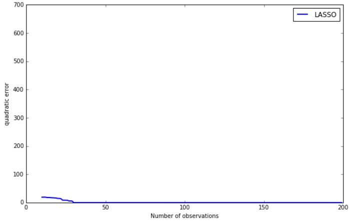

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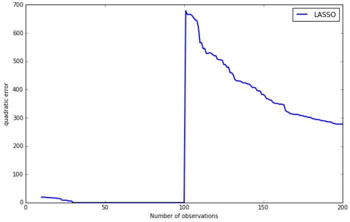

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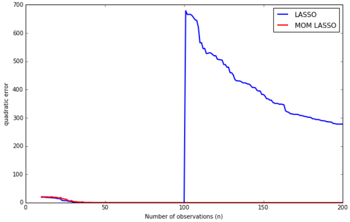

Test of robustness in classification

Dataset made of :

- 600 informative data $\overset{i.i.d.}{\sim}$ (Y, X) s.t. $\mathcal{L}(X|Y=1) = \mathcal{N}((1,1), 1.4I)$, $\mathcal{L}(X|Y=-1) = \mathcal{N}((-1,-1), 1.4I)$ and $\mathbb{P}(Y=1) = \mathbb{P}(Y=-1)$.

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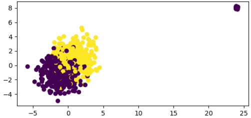
- 30 outliers data in the top corner: $Y = -1$ and $X \sim \mathcal{N}((24, 8), 0.1)$



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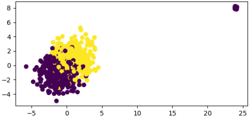
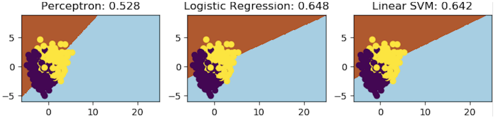


Classical procedures (Perceptron, Logistic regression, SVM):



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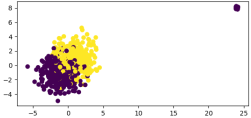

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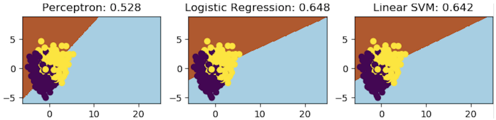
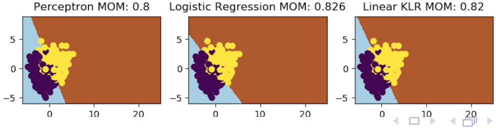


Their MOM (Median Of Means) version:



Robust statistics : motivations

- Huge datasets are likely to be corrupted by outliers
- heavy-tailed data are common in practice (like in finance)
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[Huber and Ronchetti, "Robust Statistics"]:

..we can act as if the X_i's are free of gross error

The leverage point problem

Construct procedures robust to outliers in the X_i's

A benchmark result: Let $(X_i, Y_i)_{i=1}^N$ be

- i.i.d. $\sim (X, Y)$
- $Y = \langle X, t^* \rangle + \zeta$ where $X \sim \mathcal{N}(0, I_{d\times d})$ and $\zeta \sim \mathcal{N}(0, \sigma^2)$ ind. of X,

then OLS $\hat{t} \in \underset{t\in\mathbb{R}^d}{\operatorname{argmin}} \sum_{i=1}^N (Y_i - \langle X_i, t \rangle)^2$ satisfies with probability at least $1 - c_0 \exp(-c_1 d)$,

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Question

Is it possible to construct an estimator satisfying the very same result when 1) the dataset is corrupted by **outliers** and 2) under **weak moment assumption**

From i.i.d. to the $\mathcal{O} \cup \mathcal{I}$ framework

Aim: (X, Y) a r.v., estimate $t^* \in \underset{t \in \mathbb{R}^d}{\operatorname{argmin}}\, \mathbb{E}(Y - \langle X, t \rangle)^2$.

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  2. $\forall i \in \mathcal{I}, t \in \mathbb{R}^d$,

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- $\forall t \in \mathbb{R}^d$, $\left\| \langle X, t \rangle \right\|_{L_2} \leq \theta_1 \left\| \langle X, t \rangle \right\|_{L_1}$
  (small ball assumption from [Koltchinskii & Mendelson])

Result for the MOM OLS

In the $\mathcal{O} \cup \mathcal{I}$ framework, the MOM OLS $\tilde{t}_d$ with number of blocks $K = d$ where

$$\tilde{t}_d \in \operatorname*{argmin}_{t \in \mathbb{R}^d} \sup_{t' \in \mathbb{R}^d} \mathrm{MOM}_{K=d}(\ell_t - \ell_{t'})$$

is such that with probability at least $1 - c_0 \exp(-c_1 d)$,

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Conclusion

It is possible to recover the same result in the $\mathcal{O} \cup \mathcal{I}$ framework as in the i.i.d. Gaussian with independent noise framework.

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$$\left.\begin{array}{c} Z_{N/K+1} \\ \vdots \end{array}\right\} \frac{1}{|B_2|}\sum_{i \in B_2} Z_i = P_{B_2} Z$$

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$$\Bigg\} \mathrm{Median}(P_{B_1} Z, \cdots, P_{B_K} Z) = MOM_K(Z)$$

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Key idea: $MOM_K(Z)$ is a subgaussian estimator of $\mathbb{E}Z$ under a L_2-moment assumption: if $\|Z\|_{L_2} < \infty$ then with probability at least $1 - c_0 \exp(-c_1 K)$,

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Adaptation to K via a Lepski's method:

- $\hat{I}_K = \left[MOM_K(Z) - \sigma\sqrt{K/N}, MOM_K(Z) + \sigma\sqrt{K/N}\right]$
- $\hat{K} = \min\left(K : \cap_{k=K}^{N} \hat{I}_k \neq \emptyset\right)$
- $\tilde{\mu} \in \cap_{k=\hat{K}}^{N} I_k$

Construction of MOM estimators III: MOM's principle

Aim: We are given:

- (X, Y), F and $f^* \in \underset{f \in F}{\text{argmin}}\ R(f)$ where $R(f) = \mathbb{E}\ell_f(X, Y)$ like $\ell_f(x, y) = (y - f(x))^2, \log(1 + e^{-yf(x)}), (1 - yf(x))_+, \rho_\kappa(y - f(x))$
- $(X_1, Y_1), \ldots, (X_N, Y_N)$ some data.

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We want to

- Estimate f^*: w.h.p. $\left\| \hat{f} - f^* \right\|_{L_2}^2 \leq \textit{rate}$
- Predict Y: w.h.p. $R(\hat{f}) \leq \inf_{f \in F} R(f) + \textit{residue}$

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Classical approach via ERM: $\hat{f} \in \underset{f \in F}{\text{argmin}}\ R_N(f)$ where

$$R_N(f) = P_N \ell_f = \frac{1}{N} \sum_{i=1}^{N} \ell_f(X_i, Y_i)$$

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Aim: We are given:

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- $(X_1, Y_1), \dots, (X_N, Y_N)$ some data.

We want to

- Estimate f^*: w.h.p. $\left\|\hat{f} - f^*\right\|_{L_2}^2 \leq rate$
- Predict Y: w.h.p. $R(\hat{f}) \leq \inf_{f \in F} R(f) + residue$

Classical approach via ERM: $\hat{f} \in \underset{f \in F}{\text{argmin}}\ R_N(f)$ where

$$R_N(f) = P_N\ell_f = \frac{1}{N}\sum_{i=1}^{N} \ell_f(X_i, Y_i)$$

Main idea

Replace the (non-robust) empirical mean $P_N\ell_f$ by a MOM $MOM_K(\ell_f)$ to estimate $R(f) = P\ell_f$

Construction of MOM estimators IV: MOM's principle

1) **MOM minimizer:** $\bar{f} \in \underset{f \in F}{\text{argmin}} \; MOM_K(\ell_f)$ where

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Minmax MOM estimator. Statistical properties I

Aims: (X, Y), estimate $f^* \in \operatorname*{argmin}_{f \in F} \mathbb{E}(Y - f(X))^2$ and predict Y

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  2. $\forall i \in \mathcal{I}, f\in F$ $$\|f(X_i)-f^*(X_i)\|_{L_2} = \|f(X)-f^*(X)\|_{L_2}$$ $$\|Y_i - f(X_i)\|_{L_2} = \|Y-f(X)\|_{L_2}$$

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  2. $\forall i \in \mathcal{I}, f \in F$ $\quad \|f(X_i) - f^*(X_i)\|_{L_2} = \|f(X) - f^*(X)\|_{L_2}$, $\quad \|Y_i - f(X_i)\|_{L_2} = \|Y - f(X)\|_{L_2}$
- $\zeta := Y - f^*(X)$, we assume that for all $f \in F$

$$\mathrm{var}(\zeta(f(X) - f^*(X))) \leq \sigma^2 \mathbb{E}(f(X) - f^*(X))^2$$

- $\forall f \in F$, $\|f(X_i) - f^*(X_i)\|_{L_2} \leq \theta_1 \|f(X_i) - f^*(X_i)\|_{L_1}$ (SBA)

Minmax MOM estimator. Statistical properties II

Two fixed points measuring the complexity of the problem:

$$r_Q(\gamma_Q) = \inf\left\{ r > 0 : \forall J \subset \mathcal{I}, |J| \geq \frac{N}{2}, \ \mathbb{E} \sup_{\substack{g \in F - f^* \\ \|g\|_{L_P^2} \leq r}} \left| \sum_{i \in J} \epsilon_i g(X_i) \right| \leqslant \gamma_Q |J| r \right\}$$

$$r_M(\gamma_M) = \inf\left\{ r > 0 : \forall J \subset \mathcal{I}, |J| \geq \frac{N}{2}, \ \mathbb{E} \sup_{\substack{g \in F - f^* \\ \|g\|_{L_P^2} \leq r}} \left| \sum_{i \in J} \epsilon_i \zeta_i g(X_i) \right| \leq \gamma_M |J| r^2 \right\}$$

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Let

$$r^* = \max\{r_Q(\gamma_Q), r_M(\gamma_M)\}.$$

$(r^*)^2$ is the minimax rate of convergence in the i.i.d. framework with Gaussian design and Gaussian noise independent of the design [L. & Mendelson].

Minmax MOM estimator. Statistical properties III

Theorem

In the $\mathcal{O} \cup \mathcal{I}$ framework. Let $K \in \left[\max(N(r^*)^2/\sigma^2, |\mathcal{O}|), N\right]$. With probability at least $1 - c_0 \exp(-c_1 K)$, the minmax MOM estimator

$$\hat{f}_K \in \underset{f \in F}{\operatorname{argmin}} \sup_{g \in F} MOM_K(\ell_f - \ell_g)$$

satisfies

$$\left\| \hat{f}_K - f^* \right\|_{L_2}^2 \leq c_3 \frac{\sigma^2 K}{N} \text{ and } R(\hat{f}_K) \leq \inf_{f \in F} R(f) + \frac{c_4 \sigma^2 K}{N}.$$

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In particular, for $K = \max(N(r^*)^2, |\mathcal{O}|)$,

$$\left\| \hat{f}_K - f^* \right\|_{L_2}^2, R(\hat{f}_K) - \inf_{f \in F} R(f) \leq c_4 \max\left((r^*)^2, \frac{\sigma^2 |\mathcal{O}|}{N}\right)$$

$= c_4 (r^*)^2$ (the minimax rate) when $\sigma^2 |\mathcal{O}| \leq N(r^*)^2$.

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(then, adaptation to K via Lepski's method).

Regularized minmax MOM estimators

$$\hat{f}_K \in \operatorname*{argmin}_{f \in F} \sup_{g \in F} MOM_K(\ell_f - \ell_g) + \lambda(\|f\| - \|g\|)$$

General results:

- sparsity oracle inequalities and sparse estimation rates (when $\|\cdot\|$ has some sparsity inducing power)

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Example: **MOM version of the LASSO:**

$$\hat{t}_K \in \underset{t\in\mathbb{R}^d}{\text{argmin}} \sup_{t'\in\mathbb{R}^d} MOM_K(\ell_t - \ell_{t'}) + \lambda_K\left(\|t\|_1 - \|t'\|_1\right)$$

where $\ell_t(x,y) = (y - \langle x, t\rangle)^2$ and

$$\lambda_K \sim \sigma\sqrt{\frac{1}{N}\log\left(\frac{\sigma^2 d}{K}\right)}$$

MOM version of the LASSO

Aim: Estimate $t^* \in \underset{t \in \mathbb{R}^d}{\operatorname{argmin}} \, \mathbb{E}(Y - \langle X, t \rangle)^2$ w.r.t. $s = \|t^*\|_0$.

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for $K = \max(s\log(d/s), |\mathcal{O}|)$. (adaptation via Lepski's method)

Algorithms

Descent methods for the MOM minimizer I

Problem: $u \in \mathbb{R}^d \to MOM_K(\ell_u)$ is not convex (in general) where

$$MOM_K(\ell_u) = Median\left(\frac{1}{|B_1|}\sum_{i\in B_1}(Y_i - \langle X_i, u\rangle)^2, \cdots, \frac{1}{|B_K|}\sum_{i\in B_K}(Y_i - \langle X_i, u\rangle)^2\right)$$

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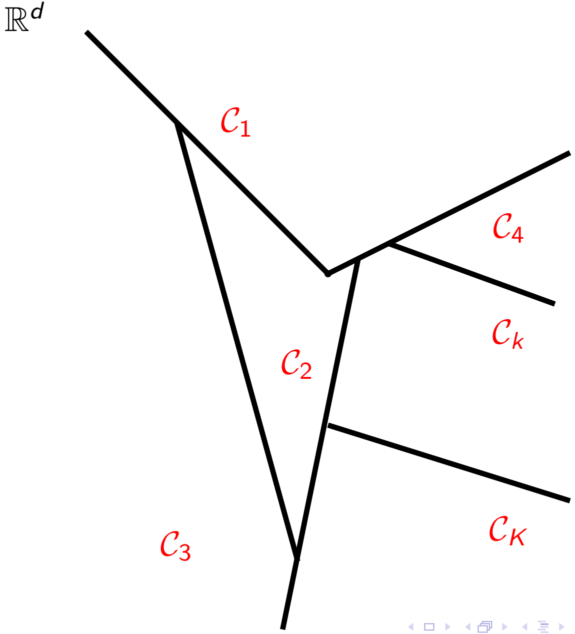
2. descent direction: $\nabla_t := \nabla(u \to P_{B_k}\ell_u)_{|u=u_t}$
3. $u_{t+1} = u_t - \eta_t \nabla_t$

Descent methods for the MOM minimizer II



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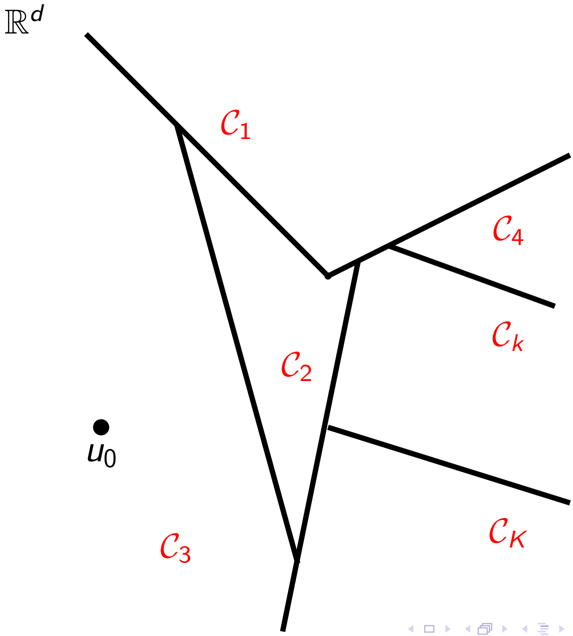
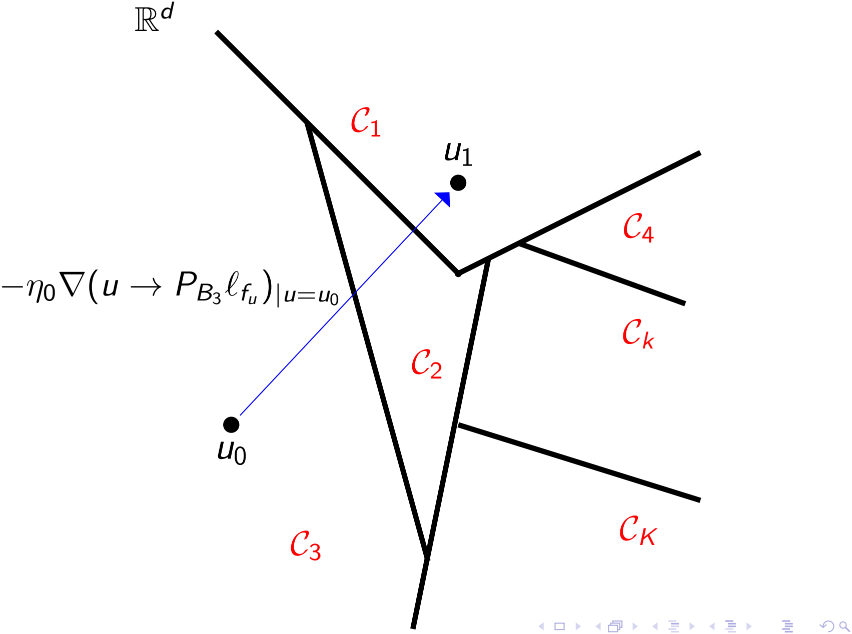


Descent methods for the MOM minimizer II



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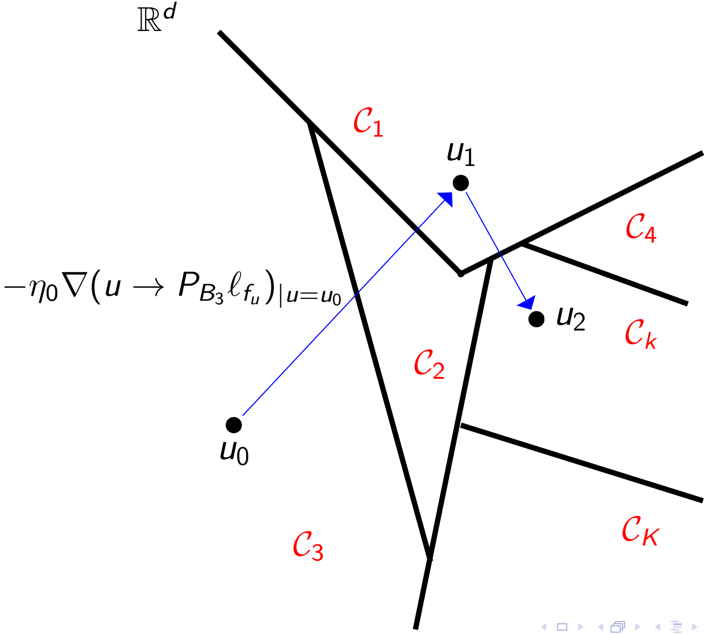

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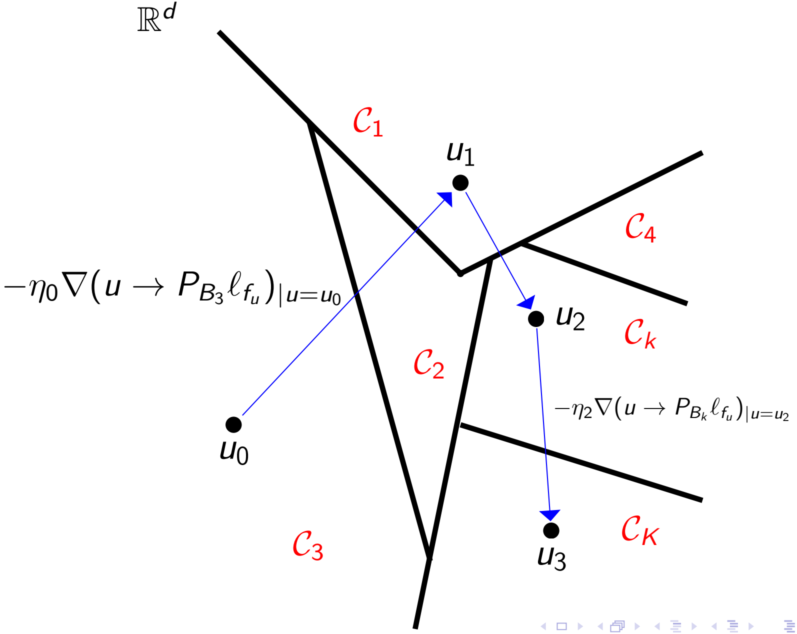

MOM GD = BGD

$$
\left.\begin{matrix}(X_1, Y_1)\\(X_2, Y_2)\\ \vdots \\ (X_{N/K}, Y_{N/K})\end{matrix}\right\} P_{B_1}\ell_{u_t}
$$

$$
\left.\begin{matrix}(X_{N/K+1}, Y_{N/K+1})\\ \vdots \end{matrix}\right\} P_{B_2}\ell_{u_t}
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MOM version of the gradient descent = **Block Gradient Descent with a particular choice of block**

1. find $k \in [K]$, s.t. $MOM_K(\ell_{u_t}) = P_{B_k}\ell_{u_t}$

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(X_2, Y_2)
$\vdots$
$(X_{N/K}, Y_{N/K})$ } $P_{B_1}\ell_{u_t}$

$(X_{N/K+1}, Y_{N/K+1})$
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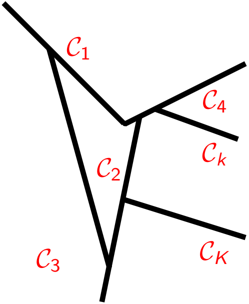
Idea: Choose the descent block according to its centrality via the median operator ("remove outliers" and closer to $\mathbb{E}\ell_{u_t}$).

Pb of local minima => Random blocks



Local minima if a cell $\mathcal{C}_k$ contains a minimum from

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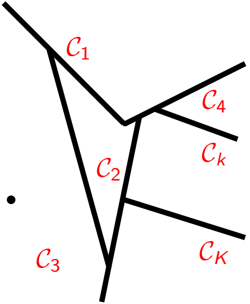

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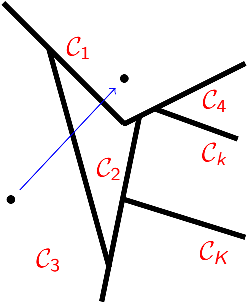

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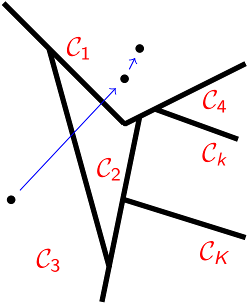

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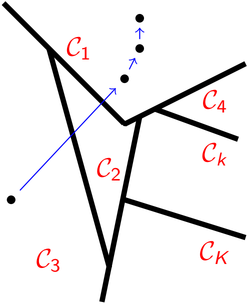

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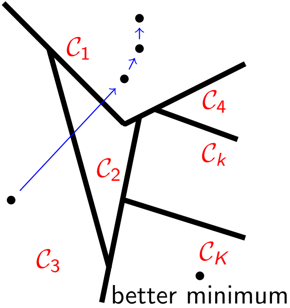

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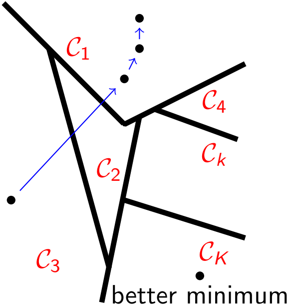

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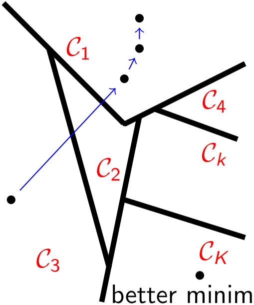

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MOM GD with random blocks = BSGD with a particular choice of the descent blocks

Convergence of the MOM GD with random blocks

Theorem

Let $\mathcal{D}_N = \{(X_i, Y_i)_{i=1}^N\}$. Assume that

1. $\|\nabla_u \ell_u(x, y)\|_2 \leq L$
2. $\hat{u} \in \underset{u \in \mathbb{R}^d}{\operatorname{argmin}} \, \mathbb{E}_{B_1 \cup \cdots \cup B_K}\left[MOM_K(\ell_u) | \mathcal{D}_N\right]$ is such that $\forall \epsilon > 0$,
$$\inf_{\|\hat{u}-u\|_2 \geq \epsilon} \left\langle \hat{u} - u, \mathbb{E}[\nabla_u \ell_u(x, y) | \mathcal{D}_N] \right\rangle > 0$$
3. $\sum_t \eta_t^2 < \infty$ and $\sum_t \eta_t = \infty$
4. for λ_d-almost all $u \in \mathbb{R}^d$, there exists an open set B such that $u \in B$ and for all partition $B_1 \cup \cdots \cup B_K$ and $v \in B$, ℓ_u and ℓ_v have the same median block.

Then, for almost all dataset $\mathcal{D}_N$,

$$\|u_T - \hat{u}\|_2 \xrightarrow[T \to \infty]{a.s} 0$$

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Idea: Alternate between ascent (for the max) and descent (for the min).

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Example for the **minmax MOM version of the LASSO:**

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where $\ell_u(x, y) = (y - \langle x, u \rangle)^2$ and $\lambda_K \sim \sigma \sqrt{(1/N) \log(\sigma^2 d/K)}$.

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Simulations: effect of random blocks on local minima

$N = 200$ i.i.d. copies of (X, Y) where

$$Y = \langle X, t^* \rangle + \zeta, \quad X \sim \mathcal{N}(0, I_{d\times d}) \quad \zeta \sim \mathcal{N}(0,1) \text{ ind. of } X$$

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Estimation error
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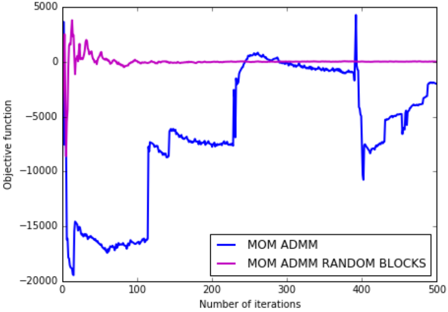

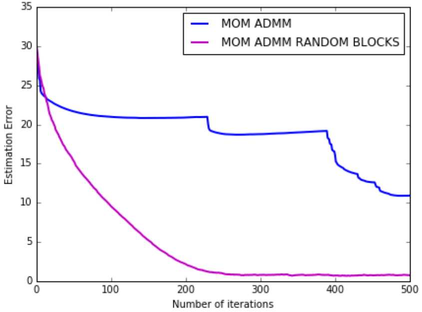

Adaptation of classical algorithms to their MOM version

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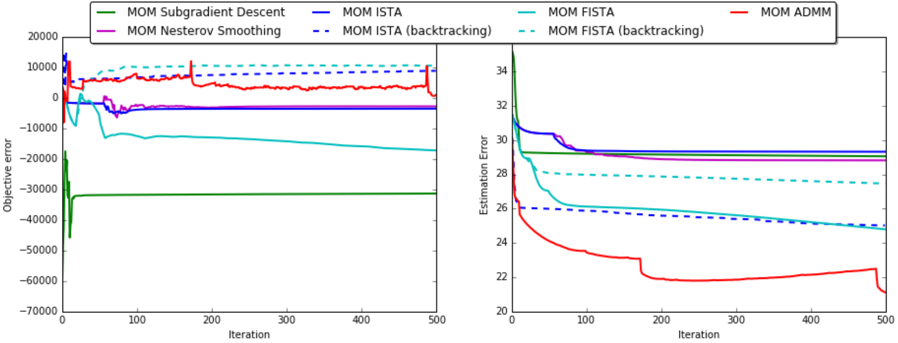
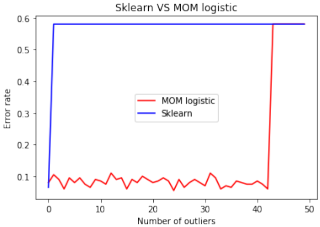


(non random blocks)

Test of robustness of minmax MOM estimators

Logistic Vs MOM logistic $N = 1000$, $d = 50$, $K = 100$



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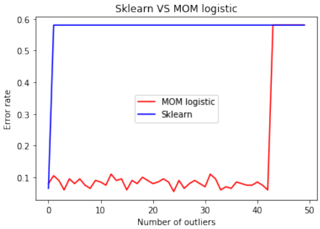
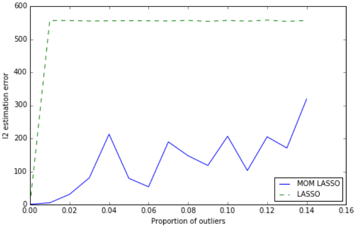


LASSO Vs MOM LASSO $N = 200$, $d = 500$, $s = 10$, adaptive choice of K and λ



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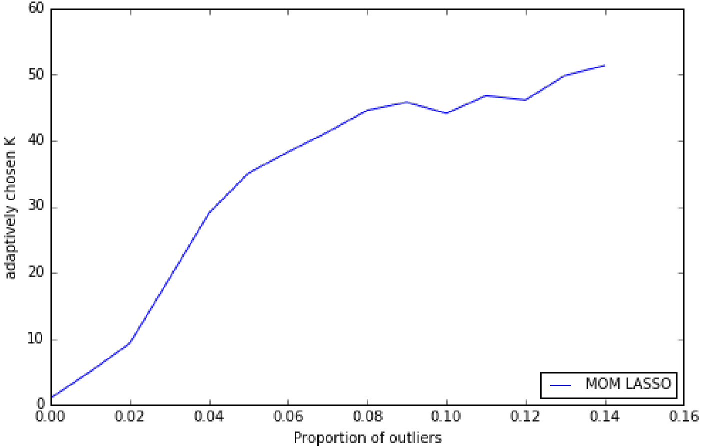
7. return $\hat{f}_{\hat{K},\hat{\lambda}}$.

Adaptively chosen number of blocks K



$\hat{K}$ increases with $|\mathcal{O}|/N$ because we need at least $K \geq 2|\mathcal{O}|$ to make MOM estimators working.

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For all $i = 1, \ldots, N$, **Score**$((X_i, Y_i))$ = number of times (X_i, Y_i) has been selected in a median block along the iterations.

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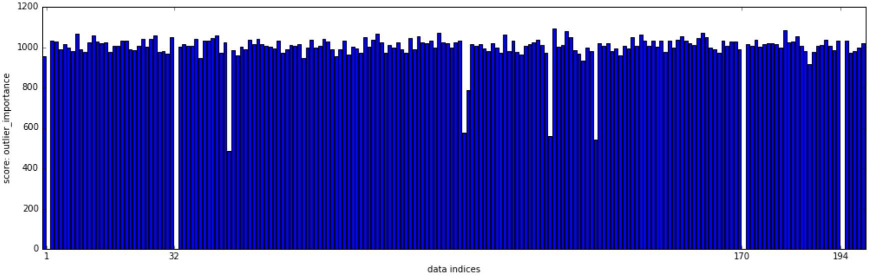

outliers are data number 1, 32, 170, 194.

Thanks!

Alternating sub-gradient descent

input : $(t_0, t'_0) \in \mathbb{R}^d \times \mathbb{R}^d$: initial point
$(\eta_p)_p, (\beta_p)_p$: two step size sequences
output: approximated solution to the min-max problem

for $t = 1, \dots, T$ **do**
    find $k \in [K]$ such that $MOM_K(\ell_{t_p} - \ell_{t'_p}) = P_{B_k}(\ell_{t_p} - \ell_{t'_p})$

$$t_{p+1} = t_p + 2\eta_p \mathbb{X}_k^\top (\mathbb{Y}_k - \mathbb{X}_k t_p) - \lambda \eta_p \mathrm{sign}(t_p)$$

    find $k \in [K]$ such that $MOM_K(\ell_{t_{p+1}} - \ell_{t'_p}) = P_{B_k}(\ell_{t_{p+1}} - \ell_{t'_p})$

$$t'_{p+1} = t'_p + 2\beta_p \mathbb{X}_k^\top (\mathbb{Y}_k - \mathbb{X}_k t'_p) - \lambda \beta_p \mathrm{sign}(t'_p)$$

end
Return (t_p, t'_p)

Alternating proximal gradient descent

input : $(t_0, t_0') \in \mathbb{R}^d \times \mathbb{R}^d$: initial point
$(\eta_k)_k, (\beta_k)_k$: two step size sequences
output: approximated solution to the min-max problem

for $t = 1, \dots, T$ **do**

find $k \in [K]$ such that $MOM_K(\ell_{t_p} - \ell_{t'_p}) = P_{B_k}(\ell_{t_p} - \ell_{t'_p})$

$$t_{p+1} = \mathrm{prox}_{\lambda\|\cdot\|_1}\left(t_p + 2\eta_k \mathbb{X}_k^\top(\mathbb{Y}_k - \mathbb{X}_k t_p)\right)$$

find $k \in [K]$ such that $MOM_K(\ell_{t_{p+1}} - \ell_{t'_p}) = P_{B_k}(\ell_{t_{p+1}} - \ell_{t'_p})$

$$t'_{p+1} = \mathrm{prox}_{\lambda\|\cdot\|_1}\left(t'_p + 2\beta_k \mathbb{X}_k^\top(\mathbb{Y}_k - \mathbb{X}_k t'_p)\right)$$

end

MOM ADMM

input : $(t_0, t'_0) \in \mathbb{R}^d \times \mathbb{R}^d$: initial point. ρ: a parameter
output: approximated solution to the min-max problem

for $t = 1, \dots, T$ **do**
  find $k \in [K]$ such that $MOM_K(\ell_{t_p} - \ell_{t'_p}) = P_{B_k}(\ell_{t_p} - \ell_{t'_p})$

$$t_{p+1} = (\mathbb{X}_k^\top \mathbb{X}_k + \rho I_{d\times d})^{-1}(\mathbb{X}_k^\top \mathbb{Y}_k + \rho z_p - u_p)$$
$$z_{p+1} = \mathrm{prox}_{\lambda\|\cdot\|_1}(t_{p+1} + u_p/\rho)$$
$$u_{p+1} = u_p + \rho(t_{p+1} - z_{p+1})$$

  find $k \in [K]$ such that $MOM_K(\ell_{t_{p+1}} - \ell_{t'_p}) = P_{B_k}(\ell_{t_{p+1}} - \ell_{t'_p})$

$$t'_{p+1} = (\mathbb{X}_k^\top \mathbb{X}_k + \rho I_{d\times d})^{-1}(\mathbb{X}_k^\top \mathbb{Y}_k + \rho z'_p - u'_p)$$
$$z'_{p+1} = \mathrm{prox}_{\lambda\|\cdot\|_1}(t'_{p+1} + u'_p/\rho)$$
$$u'_{p+1} = u'_p + \rho(t'_{p+1} - z'_{p+1})$$

end
Return (t_p, t'_p)